DEM Modeling: Lecture 13

3D Rotations

3D Rotations

- Newton's 2nd Law:

$$\frac{d}{dt}\left(\underline{\underline{\mathbf{I}}}\cdot\boldsymbol{\omega}\right)^G = \mathbf{T}^G$$

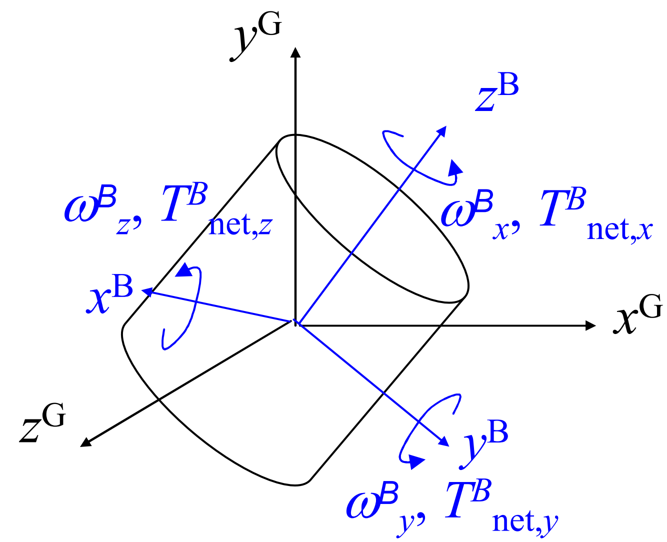
- Easiest to put the rotational eqns of motion in a body-fixed frame of reference (FOR) aligned with the particle's principle axes
  - in a body-fixed FOR, the moments of inertia don't change due to changes in orientation
  - aka Euler's rotational eqns of motion

$$\dot{\boldsymbol{\omega}}^B = \underline{\underline{\mathbf{I}}}^{-1}\cdot\left[\mathbf{T}^B - \boldsymbol{\omega}^B\times\left(\underline{\underline{\mathbf{I}}}\cdot\boldsymbol{\omega}^B\right)\right]$$

$$\dot{\omega}_x^B = \frac{1}{I_{xx}}\left[T_x^B + \omega_y^B\omega_z^B\left(I_{yy} - I_{zz}\right)\right]$$

$$\dot{\omega}_y^B = \frac{1}{I_{yy}}\left[T_y^B + \omega_z^B\omega_x^B\left(I_{zz} - I_{xx}\right)\right]$$

$$\dot{\omega}_z^B = \frac{1}{I_{zz}}\left[T_z^B + \omega_x^B\omega_y^B\left(I_{xx} - I_{yy}\right)\right]$$

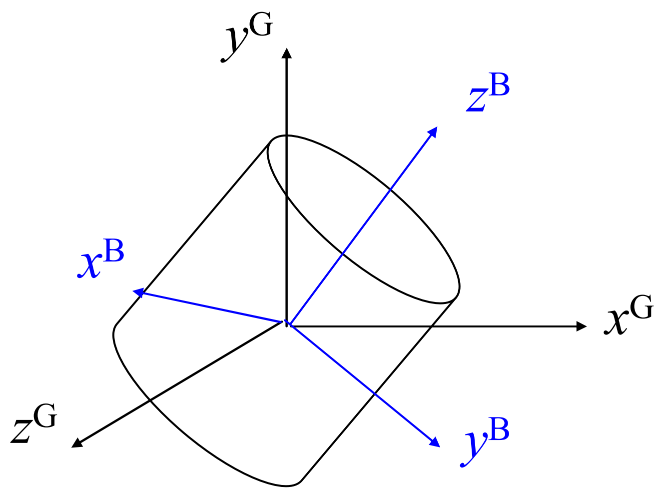


where

- I_{xx}, I_{yy}, and I_{zz} are the principle moments of inertia for the particle
- ω^B is the rotational speed of the particle in a body-fixed FOR
- $\mathbf{T}^B$ is the net torque acting on the particle in the body-fixed FOR

3D Rotations…

- How should we describe the orientation of an object in 3D space?

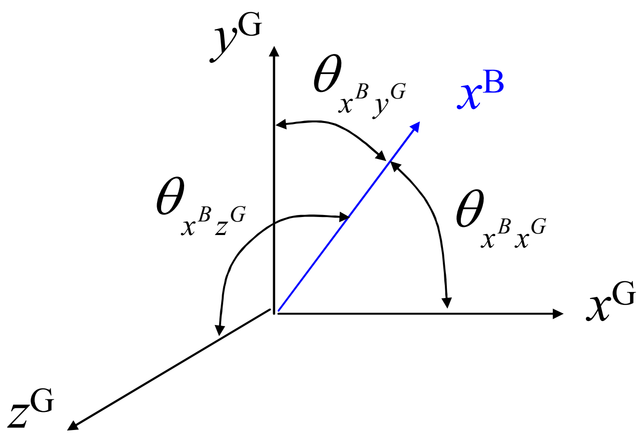


superscript "B" = body-fixed FOR
superscript "G" = global FOR

- Three common methods
  - direction cosines
  - Euler angles
  - quaternions

Direction Cosines

- Use the cosines of the angles that the body-fixed axes make with respect to the global axes to describe the orientation of the object



$$\hat{\mathbf{e}}_x^B = \hat{\mathbf{e}}_x^G \cos\theta_{x^B x^G} + \hat{\mathbf{e}}_y^G \cos\theta_{x^B y^G} + \hat{\mathbf{e}}_z^G \cos\theta_{x^B z^G}$$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x^B \\ \hat{\mathbf{e}}_y^B \\ \hat{\mathbf{e}}_z^B \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta_{x^B x^G} & \cos\theta_{x^B y^G} & \cos\theta_{x^B z^G} \\ \cos\theta_{y^B x^G} & \cos\theta_{y^B y^G} & \cos\theta_{y^B z^G} \\ \cos\theta_{z^B x^G} & \cos\theta_{z^B y^G} & \cos\theta_{z^B z^G} \end{bmatrix}}_{=R^{BG}} \begin{Bmatrix} \hat{\mathbf{e}}_x^G \\ \hat{\mathbf{e}}_y^G \\ \hat{\mathbf{e}}_z^G \end{Bmatrix}$$

$R^{BG} \equiv$ rotation matrix from the Global to the Body FOR

Direction Cosines…

- The rotation matrix R^{BG} is orthonormal
  - the basis vectors forming the rows and columns of the matrix are mutually perpendicular
  - the magnitude of the basis vectors formed by these rows and columns have a magnitude of one
  - $\Rightarrow R^{GB} = (R^{BG})^{-1} = (R^{BG})^{T}$

$$\begin{Bmatrix} \hat{\mathbf{e}}_x^G \\ \hat{\mathbf{e}}_y^G \\ \hat{\mathbf{e}}_z^G \end{Bmatrix} = R^{GB} \begin{Bmatrix} \hat{\mathbf{e}}_x^B \\ \hat{\mathbf{e}}_y^B \\ \hat{\mathbf{e}}_z^B \end{Bmatrix}$$

  - although the matrix R^{GB} has nine terms, only three are independent

Direction Cosines…

- *Example*
  - The unit vectors for the body-fixed FOR written in terms of the global FOR are:

$$\hat{\mathbf{e}}_x^B = \hat{\mathbf{e}}_y^G$$

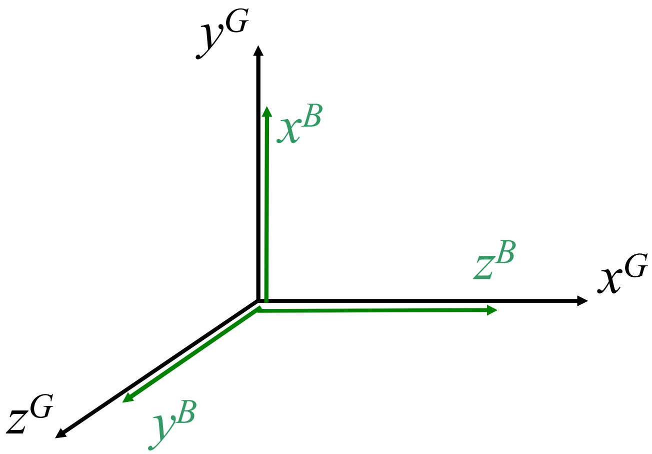
$$\hat{\mathbf{e}}_y^B = \hat{\mathbf{e}}_z^G$$

$$\hat{\mathbf{e}}_z^B = \hat{\mathbf{e}}_x^G$$

  - Sketch both the body-fixed and global FOR axes. Let the global and body-fixed FORs share the same origin.
  - Write the rotation matrix for the transformation from the body-fixed FOR to the global FOR.
  - Express the vector $\mathbf{v}^B = (0, 1, 0)$ which is given in the body-fixed FOR in terms of the global FOR using the rotation matrix determined in the previous part.

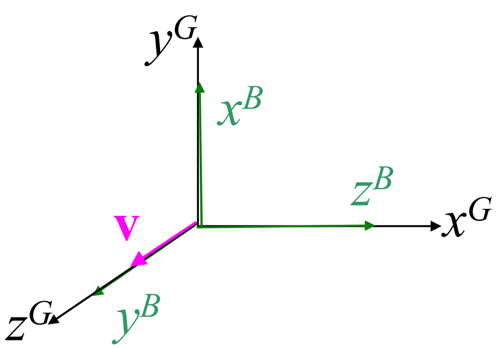
Direction Cosines…

- *Solution*



$$R^{BG} = \begin{bmatrix} \cos\theta_{x^B x^G} & \cos\theta_{x^B y^G} & \cos\theta_{x^B z^G} \\ \cos\theta_{y^B x^G} & \cos\theta_{y^B y^G} & \cos\theta_{y^B z^G} \\ \cos\theta_{z^B x^G} & \cos\theta_{z^B y^G} & \cos\theta_{z^B z^G} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R^{GB} = \left(R^{BG}\right)^{-1} = \left(R^{BG}\right)^{T} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



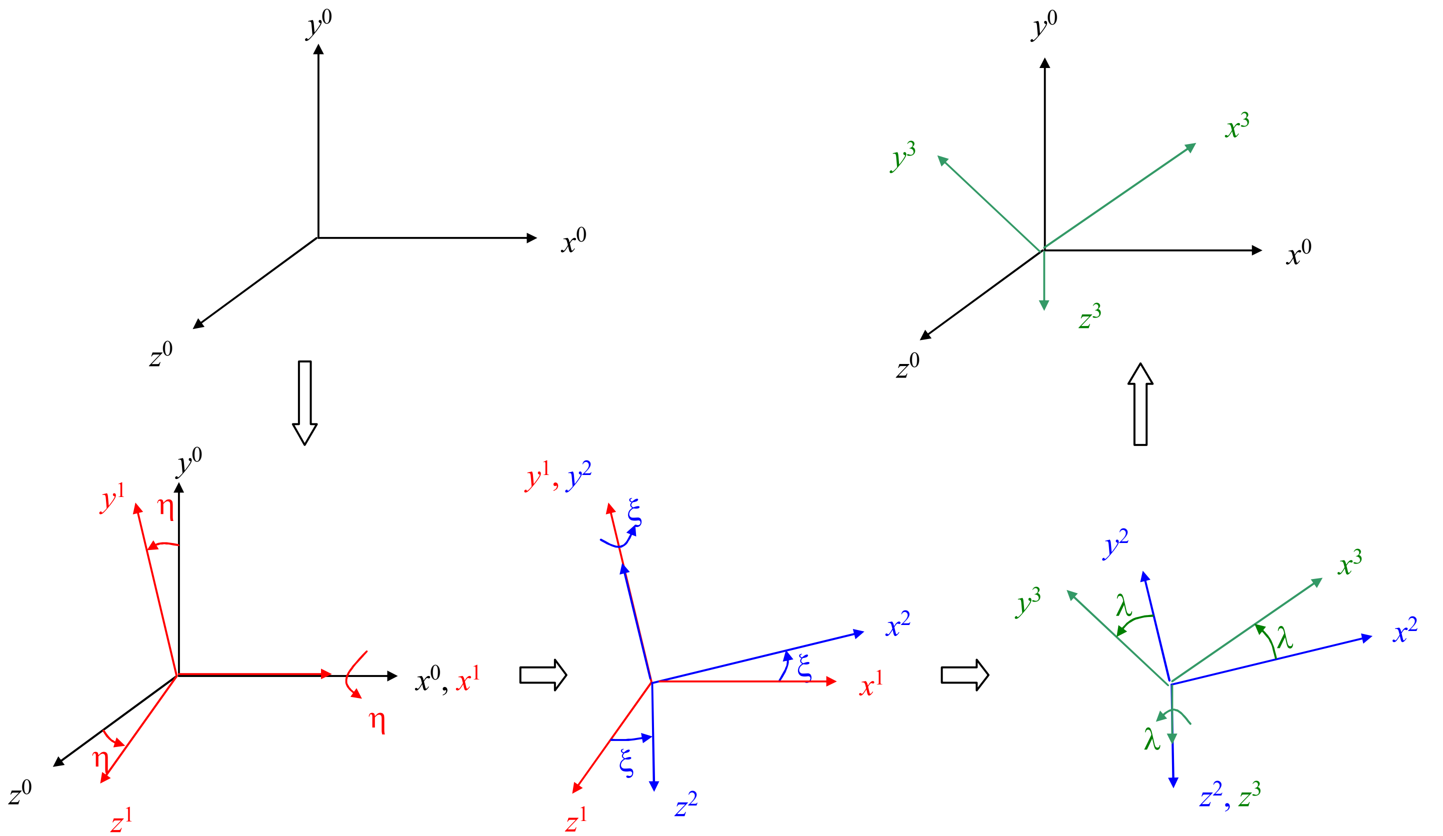
$$\mathbf{v}^G = R^{GB}\mathbf{v}^B$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\therefore \mathbf{v}^G = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Euler Angles

- Decompose the orientation into three successive rotations about various coordinate axes (*e.g.* the 1-2-3 sequence shown below)



Euler Angles…

- The full sequence of rotations may be expressed as a single rotation matrix

$$\mathbf{a}^1 = R^{10}\mathbf{a}^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & \sin\eta \\ 0 & -\sin\eta & \cos\eta \end{bmatrix}\mathbf{a}^0$$

$$\mathbf{a}^2 = R^{21}\mathbf{a}^1 = \begin{bmatrix} \cos\xi & 0 & -\sin\xi \\ 0 & 1 & 0 \\ \sin\xi & 0 & \cos\xi \end{bmatrix}\mathbf{a}^1$$

$$\mathbf{a}^3 = R^{32}\mathbf{a}^2 = \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{a}^2$$

$$\mathbf{a}^3 = R^{30}\mathbf{a}^0 = R^{32}R^{21}R^{10}\mathbf{a}^0$$

$$R^{30} = \begin{bmatrix} c\xi c\lambda & c\eta s\lambda + s\eta s\xi c\lambda & s\eta s\lambda - c\eta s\xi c\lambda \\ -c\xi s\lambda & c\eta c\lambda - s\eta s\xi s\lambda & s\eta c\lambda + c\eta s\xi s\lambda \\ s\xi & -s\eta c\xi & c\eta c\xi \end{bmatrix}$$

Euler Angles…

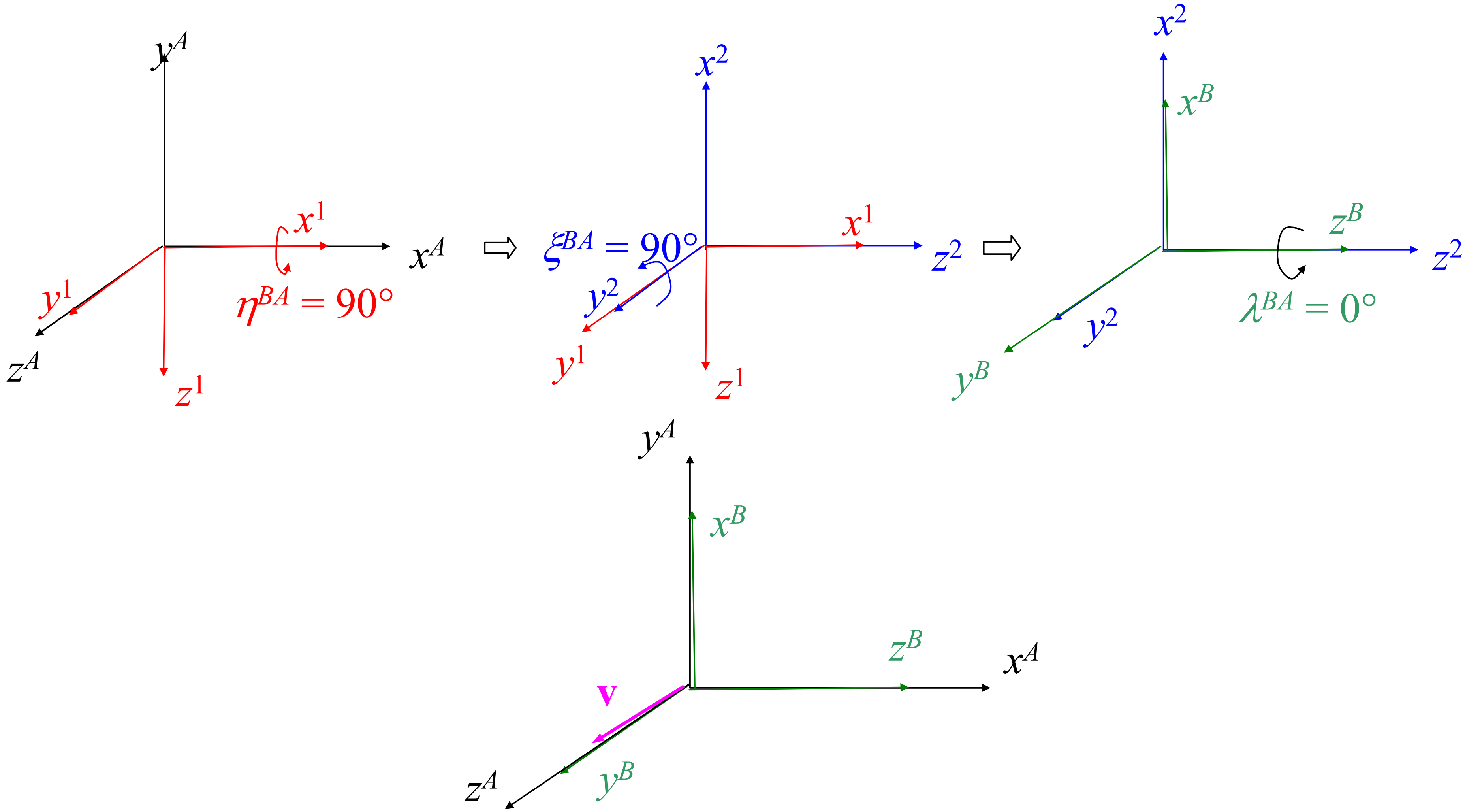
- There are twelve possible Euler angle sequences that could be used:. 1-2-1 (*e.g.* $x^0y^1x^2$), 1-3-1, 2-1-2, 2-3-2, 3-1-3, 3-2-3, 1-2-3, 2-3-1, 3-1-2, 3-2-1, 2-1-3, and 1-3-2.
- The rotation matrix is orthonormal.
- The Euler angles used to achieve a final orientation are not unique. For example, the rotation matrix is identical for $(\eta, \xi, \lambda) = (135^\circ, 135^\circ, 135^\circ)$ and $(\eta, \xi, \lambda) = (-45^\circ, 45^\circ, -45^\circ)$.

Euler Angles…

- *Example*
  - A vector $\mathbf{v}^B = (0, 1, 0)$ is expressed in FOR B. The Euler angles for converting from FOR A to FOR B are $(\eta, \xi, \lambda)^{BA} = (90°, 90°, 0°)$. Sketch FORs A and B and the vector **v**. Find vector **v**'s components in FOR A.

Euler Angles…

- *Solution*



Euler Angles…

- *Solution...*

$$\mathbf{v}^B = R^{BA}\mathbf{v}^A \Rightarrow \mathbf{v}^A = \left(R^{BA}\right)^{-1}\mathbf{v}^B = \left(R^{BA}\right)^T\mathbf{v}^B$$

$$R^{BA} = \begin{bmatrix} c\xi^{BA}c\lambda^{BA} & c\eta^{BA}s\lambda^{BA} + s\eta^{BA}s\xi^{BA}c\lambda^{BA} & s\eta^{BA}s\lambda^{BA} - c\eta^{BA}s\xi^{BA}c\lambda^{BA} \\ -c\xi^{BA}s\lambda^{BA} & c\eta^{BA}c\lambda^{BA} - s\eta^{BA}s\xi^{BA}s\lambda^{BA} & s\eta^{BA}c\lambda^{BA} + c\eta^{BA}s\xi^{BA}s\lambda^{BA} \\ s\xi^{BA} & -s\eta^{BA}c\xi^{BA} & c\eta^{BA}c\xi^{BA} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}^A = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{=\left(R^{BA}\right)^T} \underbrace{\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}}_{=\mathbf{v}^B}$$

$$\therefore \mathbf{v}^A = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}^A = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Euler Angles…

- Euler angle rate of change
  - consider a small rotation about each of the Euler angle sequence axes

$$\Delta\boldsymbol{\theta} = \Delta\eta\hat{\mathbf{e}}_x^0 + \Delta\xi\hat{\mathbf{e}}_y^1 + \Delta\lambda\hat{\mathbf{e}}_z^2$$

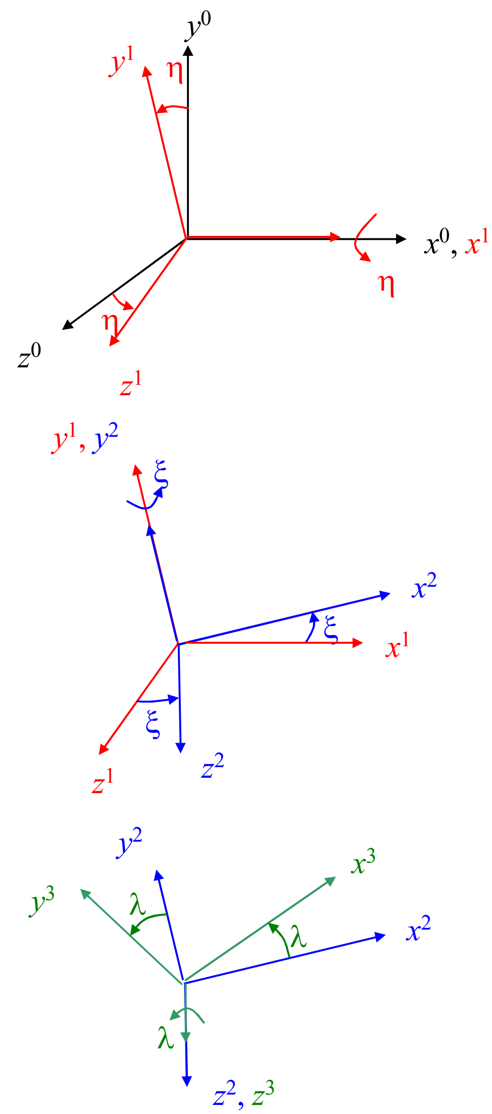
  - express the rate at which the rotation, $\Delta\theta$, occurs in terms of the body-fixed FOR (frame 3)

$$\hat{\mathbf{e}}_z^2 = \hat{\mathbf{e}}_z^3$$

$$\hat{\mathbf{e}}_y^1 = \hat{\mathbf{e}}_y^2 = \sin\lambda\hat{\mathbf{e}}_x^3 + \cos\lambda\hat{\mathbf{e}}_y^3$$

$$\hat{\mathbf{e}}_x^0 = \hat{\mathbf{e}}_x^1 = \cos\xi\hat{\mathbf{e}}_x^2 + \sin\xi\hat{\mathbf{e}}_z^2$$

$$\hat{\mathbf{e}}_x^2 = \cos\lambda\hat{\mathbf{e}}_x^3 - \sin\lambda\hat{\mathbf{e}}_y^3$$



Euler Angles…

- Euler angle rate of change…

$$\Delta\boldsymbol{\theta}^3 = \left(\Delta\eta\cos\xi\cos\lambda + \Delta\xi\sin\lambda\right)\hat{\mathbf{e}}_x^3 + \left(-\Delta\eta\cos\xi\sin\lambda + \Delta\xi\cos\lambda\right)\hat{\mathbf{e}}_y^3 + \left(\Delta\eta\sin\xi + \Delta\lambda\right)\hat{\mathbf{e}}_z^3$$

$$\underbrace{\frac{d\boldsymbol{\theta}^3}{dt}}_{=\boldsymbol{\omega}^3} = \left(\underbrace{\frac{d\eta}{dt}}_{=\dot{\eta}}\cos\xi\cos\lambda + \underbrace{\frac{d\xi}{dt}}_{=\dot{\xi}}\sin\lambda\right)\hat{\mathbf{e}}_x^3 + \left(-\underbrace{\frac{d\eta}{dt}}_{=\dot{\eta}}\cos\xi\sin\lambda + \underbrace{\frac{d\xi}{dt}}_{=\dot{\xi}}\cos\lambda\right)\hat{\mathbf{e}}_y^3 + \left(\underbrace{\frac{d\eta}{dt}}_{=\dot{\eta}}\sin\xi + \underbrace{\frac{d\lambda}{dt}}_{=\dot{\lambda}}\right)\hat{\mathbf{e}}_z^3$$

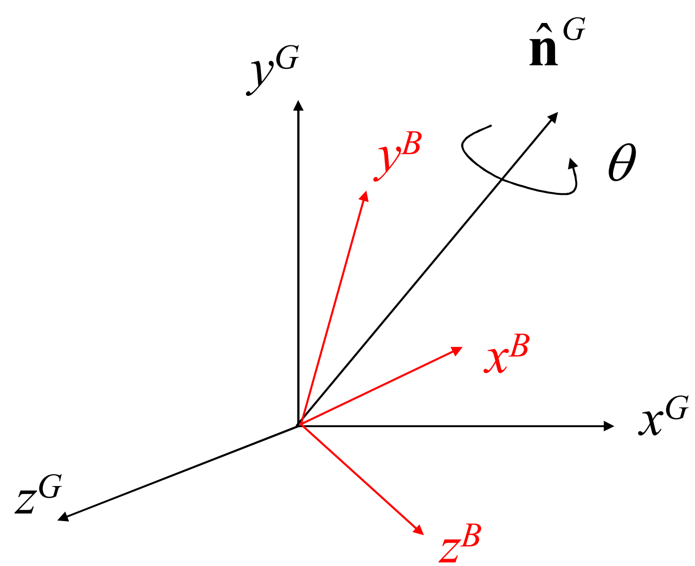
$$\begin{Bmatrix}\omega_x^3\\ \omega_y^3\\ \omega_z^3\end{Bmatrix} = \begin{bmatrix}\cos\xi\cos\lambda & \sin\lambda & 0\\ -\cos\xi\sin\lambda & \cos\lambda & 0\\ \sin\xi & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot{\eta}\\ \dot{\xi}\\ \dot{\lambda}\end{Bmatrix}$$

$$\therefore \begin{Bmatrix}\dot{\eta}\\ \dot{\xi}\\ \dot{\lambda}\end{Bmatrix} = \begin{bmatrix}\cos\lambda/\cos\xi & -\sin\lambda/\cos\xi & 0\\ \sin\lambda & \cos\lambda & 0\\ -\tan\xi\cos\lambda & \tan\xi\sin\lambda & 1\end{bmatrix}\begin{Bmatrix}\omega_x^3\\ \omega_y^3\\ \omega_z^3\end{Bmatrix}$$

- if $\xi \to \pi/2$ or $3\pi/2$, then $\cos\xi \to 0$
- The Euler angle formulation is not well posed for all Euler angles.
- Other Euler angle sequences (*e.g.* 1-2-1) will also have singularities at particular angles.
- This behavior is sometimes referred to as “gimbal lock.”

Quaternions

- Describe an orientation using a single rotation about a unit vector



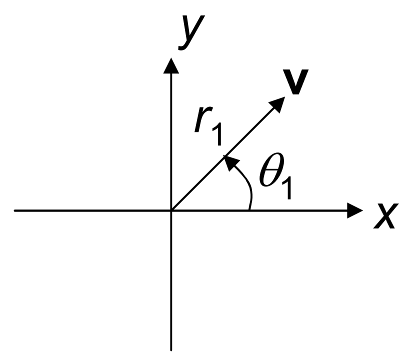
- Unlike direction cosines and Euler angles, a quaternion uses four quantities to describe the orientation: (θ, n_x^G, n_y^G, n_z^G). One of the quantities is not independent.
- An Euler angle rotation sequence can be thought of as a sequence of three quaternion rotations.

Quaternions…

- Consider a complex number

Let the 2D vector **v** be represented by the complex number z_1

$$\mathbf{v} = z_1 = (x_1, y_1) = (r_1 \cos\theta_1, r_1 \sin\theta_1)$$



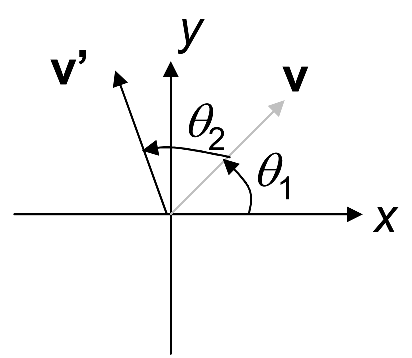
$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

We can rotate the vector **v** about the origin by an angle θ_2 by multiplying z_1 by the complex number z_2 as given below.

$$z_2 = (\cos\theta_2, \sin\theta_2)$$

$$\mathbf{v}' = z_1 z_2 = (r_1 \cos\theta_1, r_1 \sin\theta_1)(\cos\theta_2, \sin\theta_2)$$
$$= (r_1 \cos(\theta_1 + \theta_2), r_1 \sin(\theta_1 + \theta_2))$$

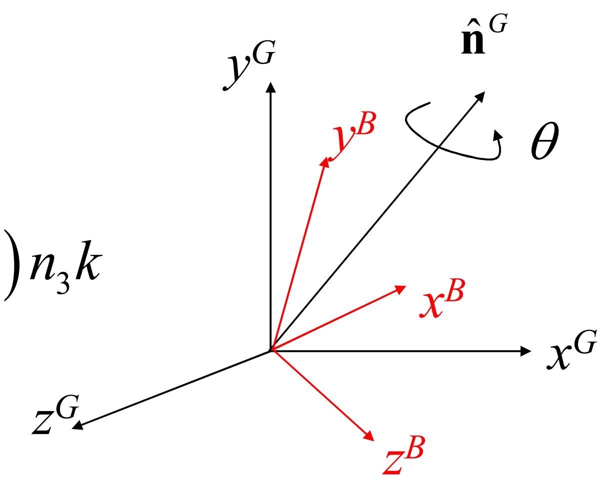


$\therefore$**2D rotations may be performed using complex numbers**

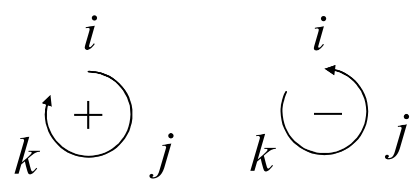
Quaternions…

- (Unit) quaternions are "hypercomplex" numbers that can be used to perform 3D rotations
  - originally proposed by Hamilton (1843)

$$\hat{\mathbf{q}} = q_0 + q_1 i + q_2 j + q_3 k$$
$$= \left[\cos\left(\tfrac{\theta}{2}\right), \hat{\mathbf{n}}\sin\left(\tfrac{\theta}{2}\right)\right]$$
$$= \cos\left(\tfrac{\theta}{2}\right) + \sin\left(\tfrac{\theta}{2}\right) n_1 i + \sin\left(\tfrac{\theta}{2}\right) n_2 j + \sin\left(\tfrac{\theta}{2}\right) n_3 k$$



$$i^2 = j^2 = k^2 = -1$$
$$ij = -ji = k$$
$$jk = -kj = i$$
$$ki = -ik = j$$



Quaternions…

- Some handy quaternion properties

$$|\mathbf{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

$$\hat{\mathbf{q}} = \frac{\mathbf{q}}{|\mathbf{q}|}$$

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

$$\mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c}$$

$$\mathbf{ab} \neq \mathbf{ba}, \quad \text{in general}$$

$$\begin{aligned}\mathbf{ab} &= [s, \mathbf{u}][t, \mathbf{v}] = [st - \mathbf{u}\cdot\mathbf{v}, s\mathbf{v} + t\mathbf{u} + \mathbf{u}\times\mathbf{v}] \\ &= (a_0 + ia_1 + ja_2 + ka_3)(b_0 + ib_1 + jb_2 + kb_3) \\ &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + i(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) + \\ &\quad j(a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1) + k(a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\end{aligned}$$

Quaternions…

- Some handy quaternion properties…

$$\mathbf{q}\mathbf{q}^{-1} = [1, \mathbf{0}]$$

$$\text{if } \mathbf{q} = [s, \mathbf{v}] = (q_0, q_1, q_2, q_3), \text{ then } \mathbf{q}^{-1} = [s, -\mathbf{v}] = (q_0, -q_1, -q_2, -q_3)$$

$$\mathbf{p}^{-1}\mathbf{q}^{-1} = (\mathbf{q}\mathbf{p})^{-1}$$

A vector $\mathbf{v}$ rotated about unit vector $\hat{\mathbf{n}}$ by an angle θ is given by:

$$[0, \mathbf{v}'] = \hat{\mathbf{q}}[0, \mathbf{v}]\hat{\mathbf{q}}^{-1} = \left[0, \mathbf{v}\cos\theta + (\hat{\mathbf{n}} \times \mathbf{v})\sin\theta + (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}(1 - \cos\theta)\right]$$

Two successive rotations (first rotate using $\hat{\mathbf{q}}_1$ then rotate using $\hat{\mathbf{q}}_2$):

$$[0, \mathbf{v}''] = (\hat{\mathbf{q}}_2\hat{\mathbf{q}}_1)[0, \mathbf{v}](\hat{\mathbf{q}}_1^{-1}\hat{\mathbf{q}}_2^{-1}) \qquad \left(\text{equivalent to } [0, \mathbf{v}''] = \hat{\mathbf{r}}[0, \mathbf{v}]\hat{\mathbf{r}}^{-1} \text{ where } \hat{\mathbf{r}} = \hat{\mathbf{q}}_2\hat{\mathbf{q}}_1\right)$$

Quaternions…

- Some handy quaternion properties…

$$\left[0,\mathbf{v}'\right]=\hat{\mathbf{q}}\left[0,\mathbf{v}\right]\hat{\mathbf{q}}^{-1}\Rightarrow\mathbf{v}'=R\mathbf{v}$$

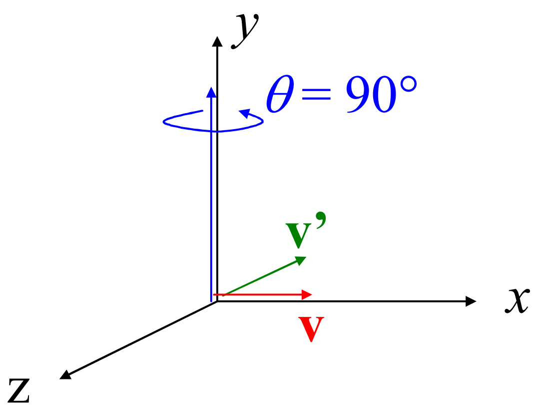
$$\text{where}\quad R=\begin{bmatrix}1-2\left(q_2^2+q_3^2\right) & 2\left(-q_0q_3+q_1q_2\right) & 2\left(q_0q_2+q_1q_3\right)\\ 2\left(q_0q_3+q_1q_2\right) & 1-2\left(q_1^2+q_3^2\right) & 2\left(-q_0q_1+q_2q_3\right)\\ 2\left(-q_0q_2+q_1q_3\right) & 2\left(q_0q_1+q_2q_3\right) & 1-2\left(q_1^2+q_2^2\right)\end{bmatrix}$$

Quaternions…

- *Example*
  - A vector **v** = (1, 0, 0) is rotated about the *y*-axis by an angle of 90º. Determine the (unit) quaternion encoding this rotation and show that the rotated vector is **v’** = (0, 0, -1). Also find the rotation matrix corresponding to the quaternion.

Quaternions…

- *Solution*



$$\hat{\mathbf{q}} = \left[\cos\left(\tfrac{\theta}{2}\right), \sin\left(\tfrac{\theta}{2}\right)\hat{\mathbf{n}}\right] \quad \text{where } \theta = 90^\circ, \hat{\mathbf{n}} = (0,1,0)$$

$$\Rightarrow \hat{\mathbf{q}} = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}, 0\right)$$

$$[0,\mathbf{v}'] = \hat{\mathbf{q}}[0,\mathbf{v}]\hat{\mathbf{q}}^{-1} = \left[0, \mathbf{v}\cos\theta + (\hat{\mathbf{n}}\times\mathbf{v})\sin\theta + (\mathbf{v}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}(1-\cos\theta)\right]$$

$$= \left[0, \mathbf{0} + (0,1,0)\times(1,0,0) + \underbrace{\{(1,0,0)\cdot(0,1,0)\}}_{=0}(0,1,0)\right]$$

$$= \left[0, (0,0,-1)\right]$$

$$\boxed{\therefore \mathbf{v}' = (0,0,-1)}$$

Quaternions…

- *Solution*

$$R = \begin{bmatrix} 1-2\left(q_2^2+q_3^2\right) & 2\left(q_1q_2-q_0q_3\right) & 2\left(q_0q_2+q_1q_3\right) \\ 2\left(q_0q_3+q_1q_2\right) & 1-2\left(q_1^2+q_3^2\right) & 2\left(-q_0q_1+q_2q_3\right) \\ 2\left(-q_0q_2+q_1q_3\right) & 2\left(q_0q_1+q_2q_3\right) & 1-2\left(q_1^2+q_2^2\right) \end{bmatrix}$$

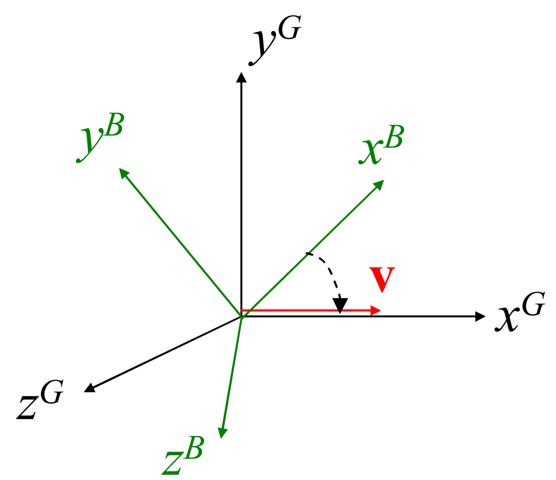
$$\boxed{\therefore R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}}$$

Note: $$\begin{Bmatrix} v_1' \\ v_2' \\ v_3' \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{=R} \underbrace{\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}}_{=\mathbf{v}} = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}$$ (same result as before)

Quaternions…

- Using quaternions to change FORs
  - Let $\mathbf{q}^{GB}$ be the unit quaternion that rotates FOR G to FOR B.
  - Since the vector **v** does not rotate with the FOR, it's as if **v** rotates in the *opposite* sense of $\mathbf{q}^{GB}$ (imagine the movement of **v** if standing in FOR B).
  - Thus, to express **v** in FOR B, rotate $\mathbf{v}^G$ using the inverse quaternion $\mathbf{q}^{BG} = (\mathbf{q}^{GB})^{-1}$.

$$\begin{aligned}\left[0, \mathbf{v}^B\right] &= \left(\hat{\mathbf{q}}^{BG}\right)\left[0, \mathbf{v}^G\right]\left(\hat{\mathbf{q}}^{BG}\right)^{-1} \\ &= \left(\hat{\mathbf{q}}^{GB}\right)^{-1}\left[0, \mathbf{v}^G\right]\left(\hat{\mathbf{q}}^{GB}\right)\end{aligned}$$



$$\mathbf{v}^B = R^{GB}\mathbf{v}^G$$

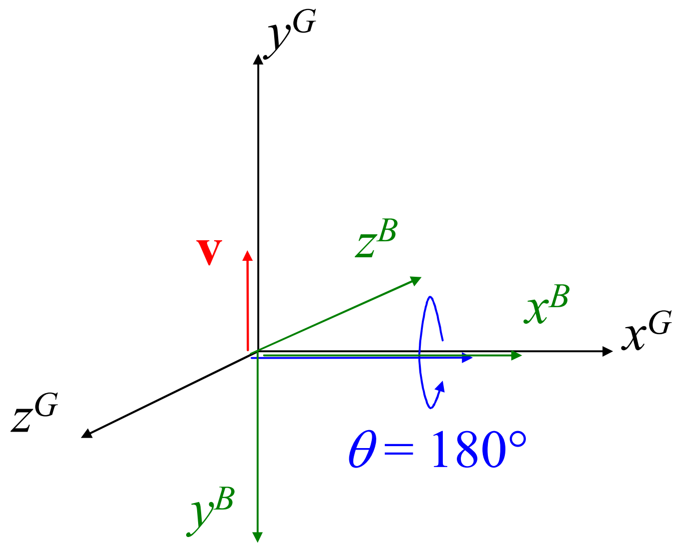
where
$$R^{GB} = \begin{bmatrix} 1-2\left[\left(q_2^{GB}\right)^2+\left(q_3^{GB}\right)^2\right] & 2\left(q_0^{GB}q_3^{GB}+q_1^{GB}q_2^{GB}\right) & 2\left(-q_0^{GB}q_2^{GB}+q_1^{GB}q_3^{GB}\right) \\ 2\left(q_1^{GB}q_2^{GB}-q_0^{GB}q_3^{GB}\right) & 1-2\left[\left(q_1^{GB}\right)^2+\left(q_3^{GB}\right)^2\right] & 2\left(q_0^{GB}q_1^{GB}+q_2^{GB}q_3^{GB}\right) \\ 2\left(q_0^{GB}q_2^{GB}+q_1^{GB}q_3^{GB}\right) & 2\left(-q_0^{GB}q_1^{GB}+q_2^{GB}q_3^{GB}\right) & 1-2\left[\left(q_1^{GB}\right)^2+\left(q_2^{GB}\right)^2\right] \end{bmatrix}$$

Quaternions…

- *Example*
  - A vector $\mathbf{v}^G = (0, 1, 0)$ is expressed in FOR G. FOR B is found by rotating FOR G about the vector $(1, 0, 0)^G$ (expressed in FOR G) by an angle of 180°. Determine the (unit) quaternion encoding the rotation from FOR G to FOR B. Express the vector **v** in FOR B.

Quaternions…

- *Solution*



$$\hat{\mathbf{q}}^{GB} = \left[\cos\left(\tfrac{\theta}{2}\right), \sin\left(\tfrac{\theta}{2}\right)\hat{\mathbf{n}}^G\right] = (0,1,0,0)$$

$$\left[0, \mathbf{v}^B\right] = \left(\hat{\mathbf{q}}^{GB}\right)^{-1}\left\{\left[0,\mathbf{v}^G\right]\left(\hat{\mathbf{q}}^{GB}\right)\right\}$$

$$= (0,-1,0,0)\left\{(0,0,1,0)(0,1,0,0)\right\}$$

$$= (0,-1,0,0)(0,0,0,-1)$$

$$= (0,0,-1,0)$$

$$\boxed{\therefore \mathbf{v}^B = (0,-1,0)}$$

Quaternions…

- Quaternion rate of change
  - consider a rotation about the body-fixed axes over a short time Δt
  - $\omega^B(t)$ is the rate of rotation about the body-fixed axes at time t
  - $\mathbf{q}^{GB}(t)$ is the unit quaternion to go from FOR B to FOR B at time t
  - the new quaternion may be found by multiplying the old quaternion by the quaternion describing the incremental rotation

$$\hat{\mathbf{q}}^{GB}(t+\Delta t) = \left[\cos\left(\frac{\left|\boldsymbol{\omega}^G(t)\right|\Delta t}{2}\right), \frac{\boldsymbol{\omega}^G(t)}{\left|\boldsymbol{\omega}^G(t)\right|}\sin\left(\frac{\left|\boldsymbol{\omega}^G(t)\right|\Delta t}{2}\right)\right]\hat{\mathbf{q}}^{GB}(t)$$

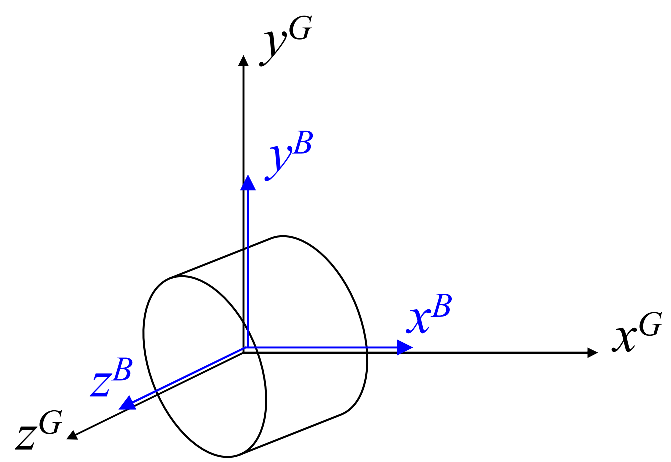
$$\boxed{\therefore \dot{\mathbf{q}}^{GB}(t) = \tfrac{1}{2}\hat{\mathbf{q}}^{GB}(t)\left[0, \boldsymbol{\omega}^B(t)\right]}$$

Note: $\left[0, \boldsymbol{\omega}^G\right] = \left(\hat{\mathbf{q}}^{GB}\right)\left[0, \boldsymbol{\omega}^B\right]\left(\hat{\mathbf{q}}^{GB}\right)^{-1}$

- there is no singularity in the quaternion rate of change (recall that there was using Euler angles)
- only unit quaternions encode rotations so the quaternion should be normalized after it has been updated to prevent numerical “drift” resulting from numerical precision errors

Quaternions…

- *Example*
  - A cylinder's body-fixed FOR (see below for the orientation of the body-fixed FOR) is found in the global FOR using the quaternion $\mathbf{q}^{GB}$ = (0, 1/sqrt(2), 0, 1/sqrt(2)). The cylinder rotates about its body-fixed axes with speed ω^{B} = (0, 0, 1). Sketch the cylinder's orientation in the global FOR and determine the rate of change of the cylinder's quaternion at this instant in time.

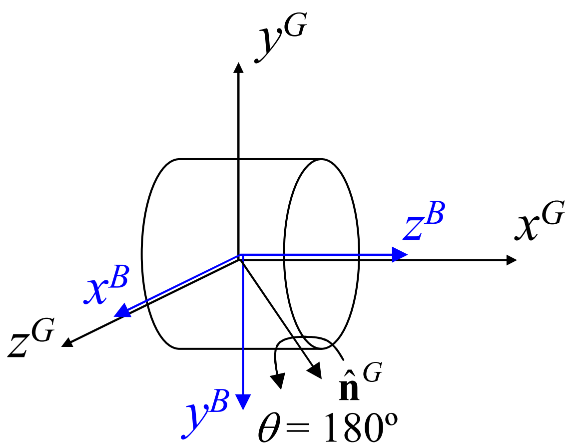


Quaternions…

- *Solution*

Note: $\theta = 180^\circ$ and $\hat{\mathbf{n}}^G = (1,0,1)$

$\Rightarrow \hat{\mathbf{q}} = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$



Determine the orientation of the body-fixed z-axis

$$\left[0, \mathbf{z}'\right] = \left(\hat{\mathbf{q}}^{GB}\right)\left[0, (0,0,1)\right]\left(\hat{\mathbf{q}}^{GB}\right)^{-1}$$

$$= \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\left\{(0,0,0,1)\left(0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)\right\}$$

$$= \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0\right)$$

$$= \left(0, \frac{1}{2} + \frac{1}{2}, 0, -\frac{1}{2} + \frac{1}{2}\right)$$

$$\therefore \mathbf{z}' = (1,0,0)$$

the body-fixed z axis points along the global x axis

Perform a similar analysis for the body-fixed x-axis

$$\left[0, \mathbf{x}'\right] = \left(\hat{\mathbf{q}}^{GB}\right)\left[0, (1,0,0)\right]\left(\hat{\mathbf{q}}^{GB}\right)^{-1}$$

$$= \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\left\{(0,1,0,0)\left(0, -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)\right\}$$

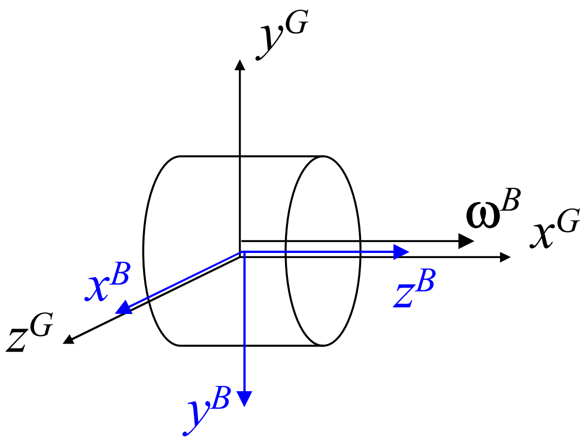
$$= \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \left(0, \frac{1}{2} - \frac{1}{2}, 0, \frac{1}{2} + \frac{1}{2}\right)$$

$$\therefore \mathbf{x}' = (0,0,1)$$

Quaternions…

- *Solution…*



$$\dot{\mathbf{q}}^{GB} = \left[0, \tfrac{1}{2}\boldsymbol{\omega}^B\right]\hat{\mathbf{q}}^{GB}$$

$$= \left(0, 0, 0, \tfrac{1}{2}\right)\left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\boxed{\therefore \dot{\mathbf{q}}^{GB} = \left(-\frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}, 0\right)}$$

Summary

- 3D orientations and rotations are more complex than their 2D counterparts
- Euler angles suffer from “gimbal lock” at particular angles – shouldn’t use for modeling rotational motion
- Quaternions have many advantages and are commonly used in DEM applications
  - don’t suffer from gimbal lock
  - easy to correct for numerical drift
  - multiple rotations can be represented by quaternion multiplication