DEM Modeling: Lecture 09
Tangential Contact Force Models

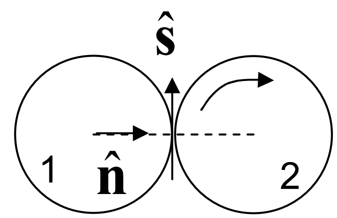
Some Preliminaries

- Most models assume that the normal contact force is independent of the tangential contact force, but the tangential contact force *is* dependent on the normal contact force.
- Tangential direction
  - perpendicular to unit normal and points in the direction of the velocity of particle 2 relative to particle 1 at the contact point

$$\Delta\dot{\mathbf{x}}_C = \dot{\mathbf{x}}_{2,C} - \dot{\mathbf{x}}_{1,C} = \left(\dot{\mathbf{x}}_2 - \dot{\mathbf{x}}_1\right) - \left(\boldsymbol{\omega}_2 \times r_2\hat{\mathbf{n}} + \boldsymbol{\omega}_1 \times r_1\hat{\mathbf{n}}\right)$$

$$\Delta\dot{\mathbf{x}}_C = \left(\Delta\dot{\mathbf{x}}_C \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \left(\Delta\dot{\mathbf{x}}_C \cdot \hat{\mathbf{s}}\right)\hat{\mathbf{s}}$$

$$\boxed{\hat{\mathbf{s}} = \frac{\Delta\dot{\mathbf{x}}_C - \left(\Delta\dot{\mathbf{x}}_C \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left|\Delta\dot{\mathbf{x}}_C - \left(\Delta\dot{\mathbf{x}}_C \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}\right|}}$$

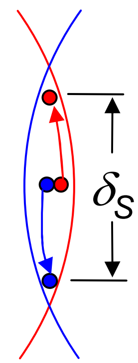


Some Preliminaries…

- Tangential velocity, $\dot{s}$

$$\dot{s} = \Delta \dot{\mathbf{x}}_C \cdot \hat{\mathbf{s}}$$

- Total tangential displacement, δ_S



$$\delta_S = \int_{t=t_0}^{t=t} \dot{s}\, dt$$

where t_0 is the time when the two particles first came into contact and t is the current time

- – the unit tangential direction may change during the contact

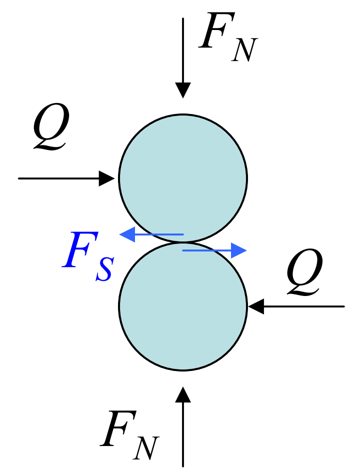
Some Preliminaries…

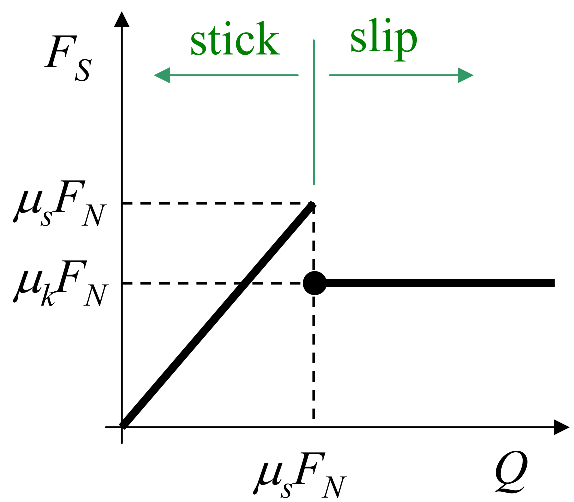
- Tangential forces are attributed to surface friction between particles.
- Friction is well modeled using Coulomb's Law of Friction,

$$F_S = \begin{cases} Q & Q < \mu_s F_N \\ \mu_k F_N & Q \geq \mu_s F_N \end{cases}$$

where μ_s is a static friction coefficient and μ_k is a kinetic (or sliding) friction coefficient

- For low shear forces there is *no* relative motion (stick)
- For high shear forces there *is* relative motion (slip)





Some Contact Mechanics

- Hertzian contact theory predicts an elliptical distribution of normal tractions over the contact region.
- Coulombic friction says shear tractions are proportional to normal tractions.
  - varying normal forces mean that some locations may slip while others stick

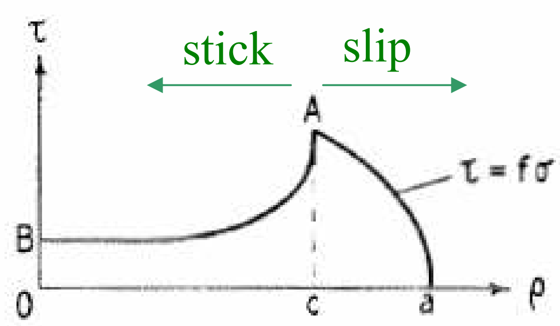


FIG. 1

From Mindlin and Deresiewicz (1953)

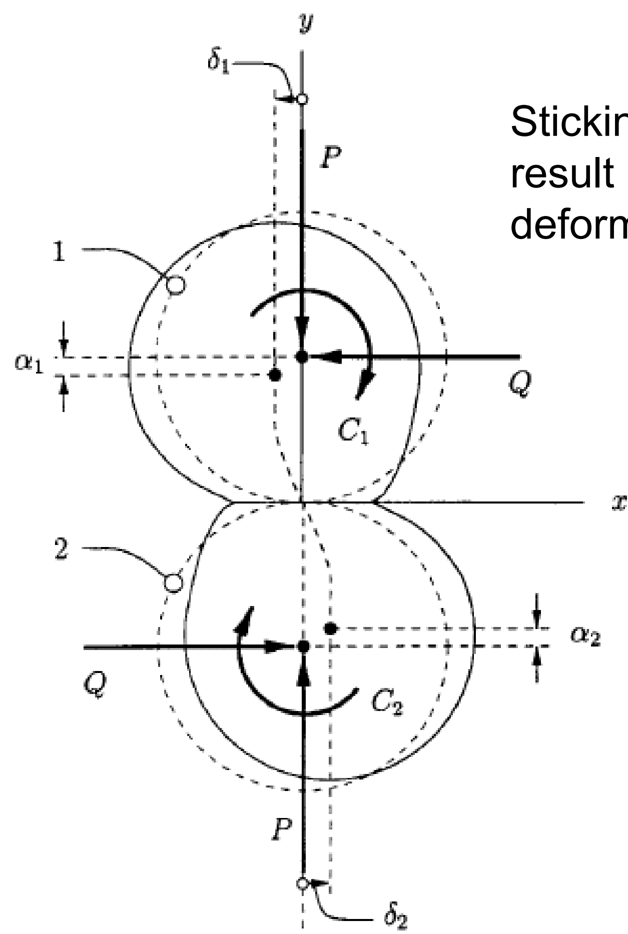


Fig. 1. Two spheres in contact and subjected to normal and tangential forces.

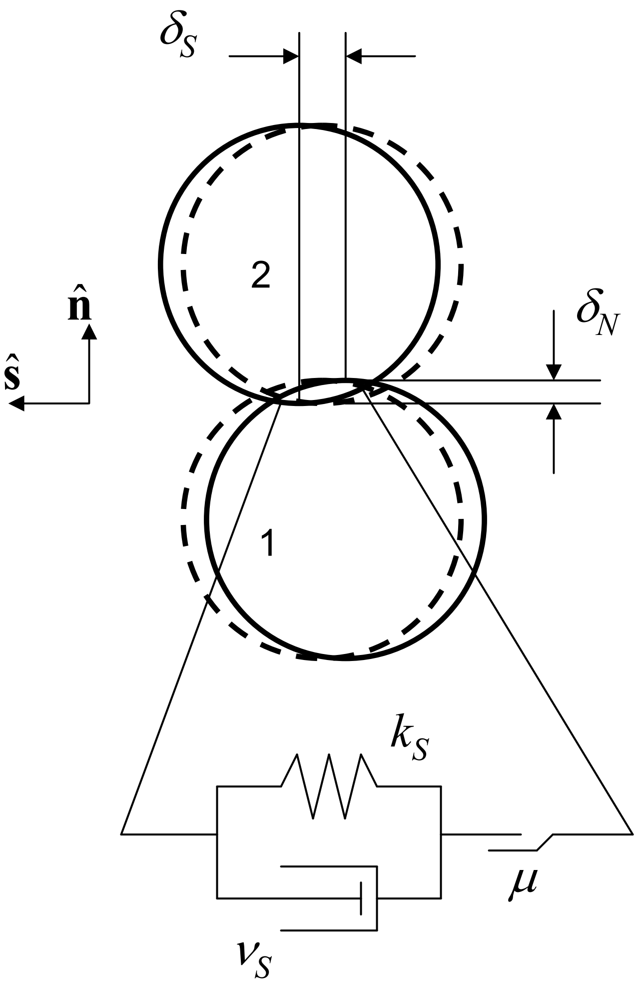
Sticking regions result in particle deformation

From Vu-Quoc and Zhang (1999)

Some Contact Mechanics…

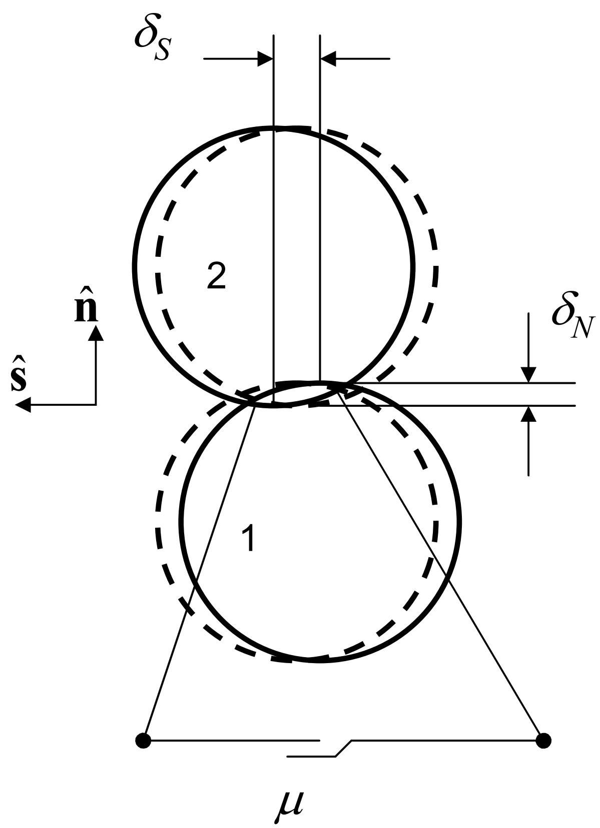
- Mindlin (1949)
  - tangential stiffness is constant for no-slip
  - shear tractions are singular at the edges of the contact region
  - non-linear stiffness for constant normal load and monotonically increasing tangential load
- Mindlin and Deresiewicz (1953)
  - hysteretic non-linear stiffness for constant oblique loading
  - continuous shear tractions with slip at contact edges
- Maw *et al*. (1976)
  - discretize contact region into annuli which are independently evaluated for slip
- Vu-Quoc *et al*. (1999)
  - many special cases (loading histories) of Mindlin and Deresiewicz (1953)

General Tangential Force Model



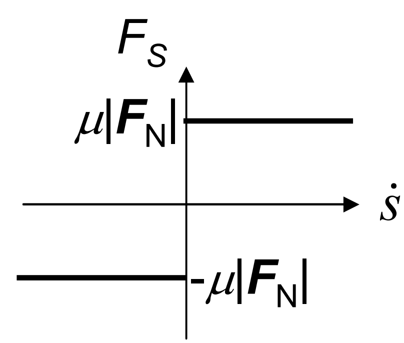
- In general, tangential force models incorporate three elements
  - spring (elastic)
  - dashpot (damping)
  - slider (friction)
- The spring constant is may be hysteretic and/or non-linear.
- The contact model parameters will be a function of both interacting materials.

Simple Coulomb Sliding



$$\mathbf{F}_{S,\text{on }1} = \mu \left|\mathbf{F}_N\right| \hat{\mathbf{s}}$$

- Widely used
- Simple model to implement
- Only incorporates sliding friction, does not account for tangential deformation $\Rightarrow$ no velocity reversal
- Force is discontinuous at $\dot{s} = 0$
  - due to dependence on $\hat{\mathbf{s}}$
- Oscillates sign as $\dot{s} \to \pm 0$ rather than $F_S \to 0$.
  - oscillation in force usually has minimal effect since the force averages to zero over time



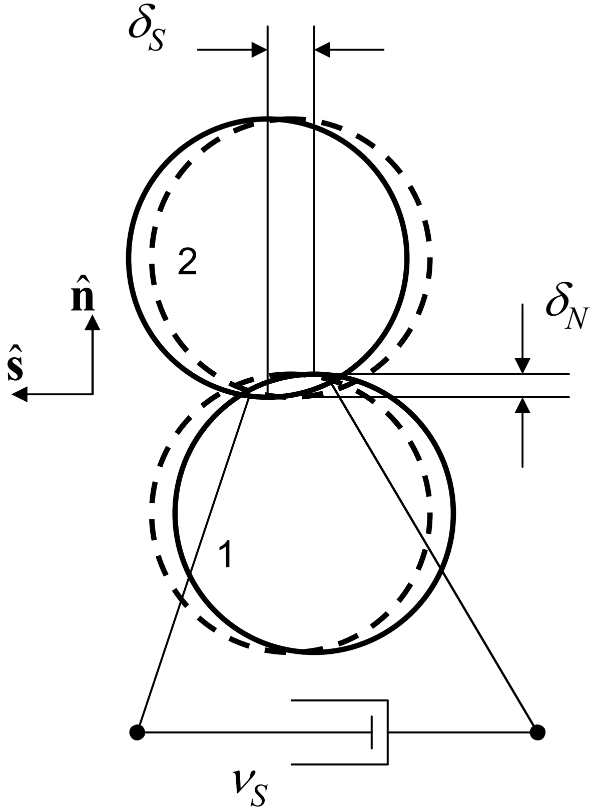
Friction Values

- Estimates of the sliding friction coefficient for various materials.

materials[1]	μ	**Reference**
soda lime glass / soda lime glass	0.092 ± 0.006	Foerster *et al*. (1994)
"fresh" glass / "fresh" glass	0.048 ± 0.006	Lorenz *et al*. (1997)
"spent" glass / "spent" glass	0.177 ± 0.020	Lorenz *et al*. (1997)
"spent" glass / "spent" glass (stationary)	0.126 ± 0.014	Lorenz *et al*. (1997)
cellulose acetate / cellulose acetate	0.25 ± 0.02	Foerster *et al*. (1994)
cellulose acetate / cellulose acetate	0.22 - 0.33	Mullier *et al*. (1991)
nylon / nylon	0.175 ± 0.1	Labous *et al*. (1997)
acrylic / acrylic	0.096 ± 0.006	Lorenz *et al*. (1997)
polystyrene / polystyrene	0.189 ± 0.009	Lorenz *et al*. (1997)
stainless steel / stainless steel	0.099 ± 0.008	Lorenz *et al*. (1997)
acrylic / aluminum plate	0.14	Mullier *et al*. (1991)
radish seeds / aluminum plate	0.19	Mullier *et al*. (1991)
"fresh" glass / aluminum plate	0.131 ± 0.007	Lorenz *et al*. (1997)
"spent" glass / aluminum plate	0.126 ± 0.009	Lorenz *et al*. (1997)
glass plate / glass plate	0.4	Beare and Bowden (1935)
glass plate / nickel plate	0.56	Beare and Bowden (1935)
glass plate / carbon plate	0.18	Beare and Bowden (1935)

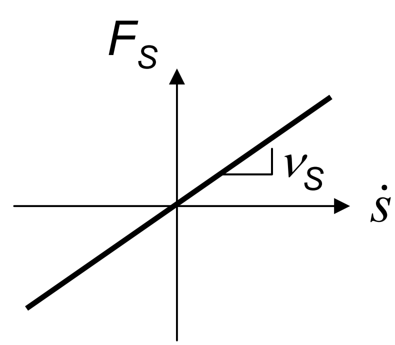
[1]interacting spheres unless otherwise noted

Viscous Damping

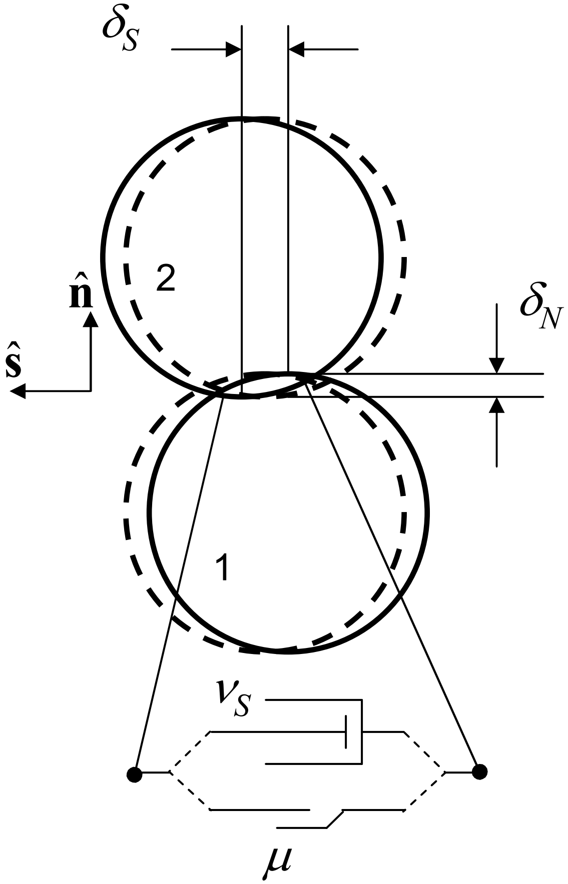


$$\mathbf{F}_{S,\text{on }1} = \nu_s \dot{s}\hat{\mathbf{s}}$$

- Not widely used
- Simple model to implement
- Model is physically justified for lubricated contacts
- No dependence on F_N
- No tangential deformation ⇒ no velocity reversal
- Force is continuous at $\dot{s} = 0$
- Poor behavior for grazing impacts since no dependence on normal force

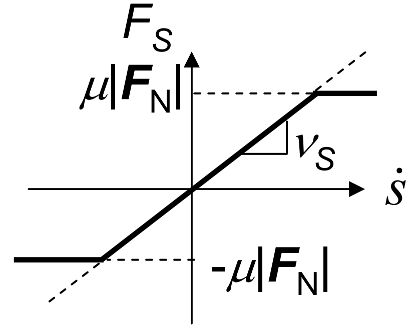


Viscous Damping with Sliding Friction

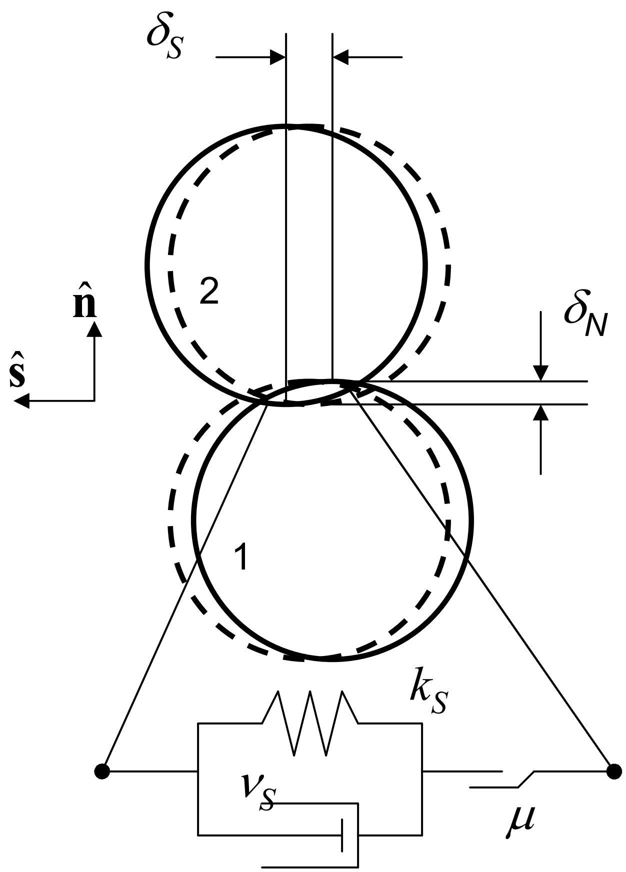


$$\mathbf{F}_{S,\text{on }1} = \min\left(\nu_S \dot{s}, \mu\left|\mathbf{F}_N\right|\right)\hat{\mathbf{s}}$$

- Not widely used
- Simple model to implement
- Model is most appropriate for lubricated contacts (refer to Ghaisas *et al*. (2004) for example)
- No tangential deformation $\Rightarrow$ no velocity reversal
- Force is continuous at $\dot{s} = 0$



Linear Damped Spring in Series with a Sliding Friction Element

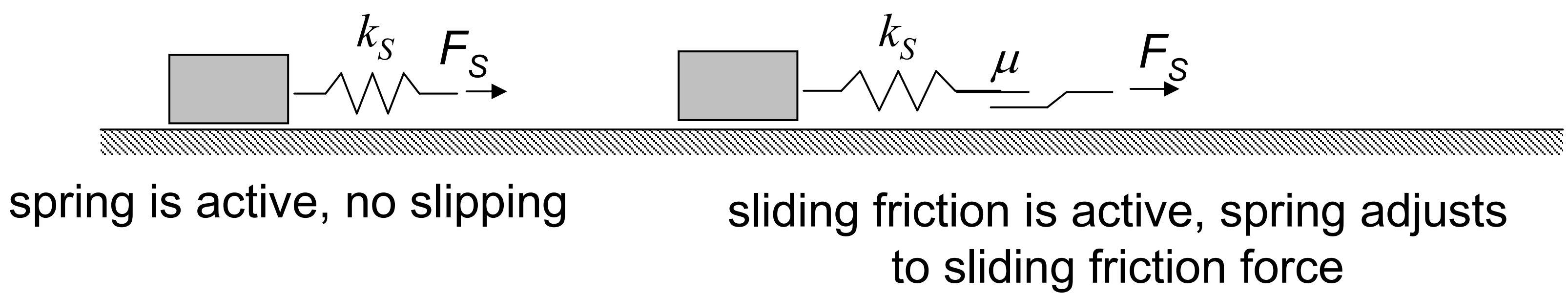


$$\mathbf{F}_{S,\text{ on }1} = \min\left(k_S\delta_S + \nu_S\dot{s}, \mu\left|\mathbf{F}_N\right|\right)\hat{\mathbf{s}}$$

- Widely used
- Moderately difficult model to implement
- Does include tangential stiffness ⇒ possible to have velocity reversal
- Dynamics of impact are governed by the ratio of the tangential to normal impact stiffnesses
  - the normal spring sets the contact duration while the tangential response is a function of the tangential spring stiffness
- If the tangential stiffness is large and the sliding friction coefficient is small, then the sliding friction element dominates during most of the contact.

Linear Damped Spring in Series with a Sliding Friction Element…

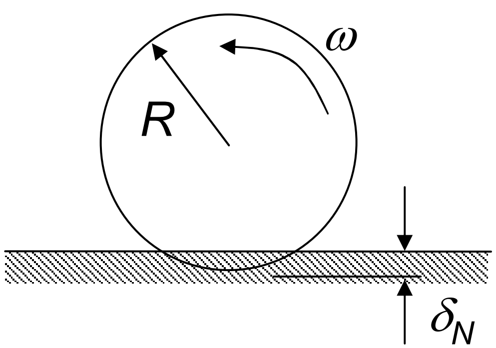
- When the sliding friction element is active, the spring extension, δ_S, is set to a value such that the spring force matches the sliding friction force.
  - the spring force will never exceed the sliding friction force since the spring will extend up to the point of slipping



$$\text{If } \mu\left|\mathbf{F}_N\right| < k_S\delta_S + \nu_S\dot{s}, \text{ then } \mathbf{F}_S = \mu\left|\mathbf{F}_N\right|\hat{\mathbf{s}} \text{ and } \delta_S = \frac{\mathbf{F}_S - \nu_S\dot{s}}{k_S}.$$

Linear Damped Spring in Series with a Sliding Friction Element…

- Consider the dynamics of a rotating, spherical particle colliding with a wall. Constrain the translational motion of the particle to only move in the vertical direction. Model the normal and tangential forces using linear springs with different spring constants.



$$\delta_N(t=0)=0 \qquad \dot{\delta}_N(t=0)=\dot{\delta}_0$$
$$\theta(t=0)=0 \qquad \dot{\theta}(t=0)=\omega$$
$$I=\tfrac{2}{5}mR^2$$

$$\left.\begin{aligned} m\ddot{\delta}_N &= -k_N\delta_N \\ I\ddot{\theta} &= -Rk_S\underbrace{R\theta}_{=\delta_S}\end{aligned}\right\}\Rightarrow$$

$$\delta_N = \dot{\delta}_0\sqrt{\frac{m}{k_N}}\sin\left(t\sqrt{\frac{k_N}{m}}\right)$$

$$\theta = \omega\sqrt{\frac{I}{R^2k_S}}\sin\left(t\sqrt{\frac{R^2k_S}{I}}\right)$$

$$\dot{\delta}_N = \dot{\delta}_0\cos\left(t\sqrt{\frac{k_N}{m}}\right)$$

$$\dot{\theta} = \omega\cos\left(t\sqrt{\frac{R^2k_S}{I}}\right)$$

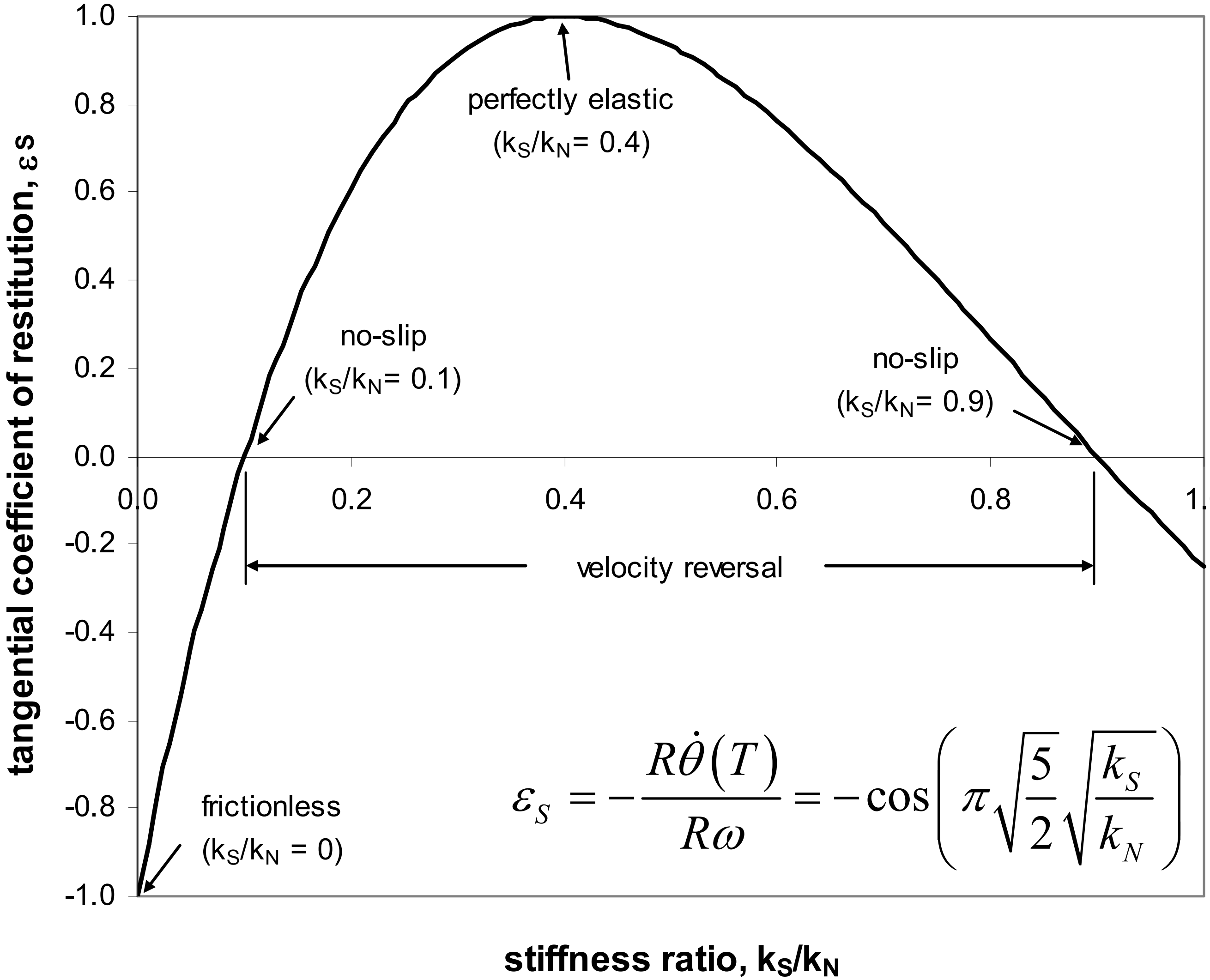
Linear Damped Spring in Series with a Sliding Friction Element…

- Contact duration is: $T = \pi\sqrt{\frac{m}{k_N}}$

- Angle and rotational speed at the end of the contact:

$$\left.\begin{aligned} \theta(T) &= \omega\sqrt{\frac{I}{R^2 k_S}}\sin\left(\pi\sqrt{\frac{m}{k_N}}\sqrt{\frac{R^2 k_S}{I}}\right) \\ \dot{\theta}(T) &= \omega\cos\left(\pi\sqrt{\frac{m}{k_N}}\sqrt{\frac{R^2 k_S}{I}}\right) \end{aligned}\right\} \Rightarrow \begin{aligned} \sqrt{\frac{5}{2}\frac{k_S}{m}}\frac{\theta(T)}{\omega} &= \sin\left(\pi\sqrt{\frac{5}{2}}\sqrt{\frac{k_S}{k_N}}\right) \\ \varepsilon_S = -\frac{R\dot{\theta}(T)}{R\omega} &= -\cos\left(\pi\sqrt{\frac{5}{2}}\sqrt{\frac{k_S}{k_N}}\right) \end{aligned}$$

Linear Damped Spring in Series with a Sliding Friction Element…



$$\varepsilon_S = -\frac{R\dot{\theta}(T)}{R\omega} = -\cos\left(\pi\sqrt{\frac{5}{2}}\sqrt{\frac{k_S}{k_N}}\right)$$

Linear Damped Spring in Series with a Sliding Friction Element…

- From the elastic solid mechanics analysis of Mindlin (1949), the stiffness ratio should be:

$$\frac{k_S}{k_N} = \frac{1-\nu}{1-\frac{1}{2}\nu} \underbrace{\Rightarrow}_{\substack{\text{most materials have} \\ \text{Poisson's ratios} \\ \text{in the range} \\ 0<\nu<\frac{1}{2}}} \frac{2}{3} < \frac{k_S}{k_N} < 1$$

Many materials have $\nu \approx 0.3 \Rightarrow \frac{k_S}{k_N} \approx 0.8$

where ν is the Poisson's ratio (assuming identical materials)

- Cundall and Strack (1979) report that the dynamics of a densely packed, quasi-statically sheared system…
  - are relatively insensitive to the stiffness ratio for small friction coefficients since sliding friction dominates through most of the contact
  - are sensitive to the stiffness ratio when the friction coefficient is large
  - larger values of k_S/k_N result in larger shear stresses since more of the load can be carried in the tangential contact direction

Linear Damped Spring in Series with a Sliding Friction Element…

- Cundall and Strack (1979) set (in an ad-hoc fashion) the tangential damping coefficient to be proportional to the tangential stiffness

$$\nu_S = \beta k_S$$

  - didn't attempt to relate β to physical parameters
  - didn't matter for them since they studied quasi-static systems; the damping only changed the rate at which the system came to rest
- It is not uncommon to set $\nu_S = 0$.

Linear Damped Spring in Series with a Sliding Friction Element…

- Tsuji *et al*. (1992) used a tangential stiffness derived analytically from the no slip solution of Mindlin (1949)

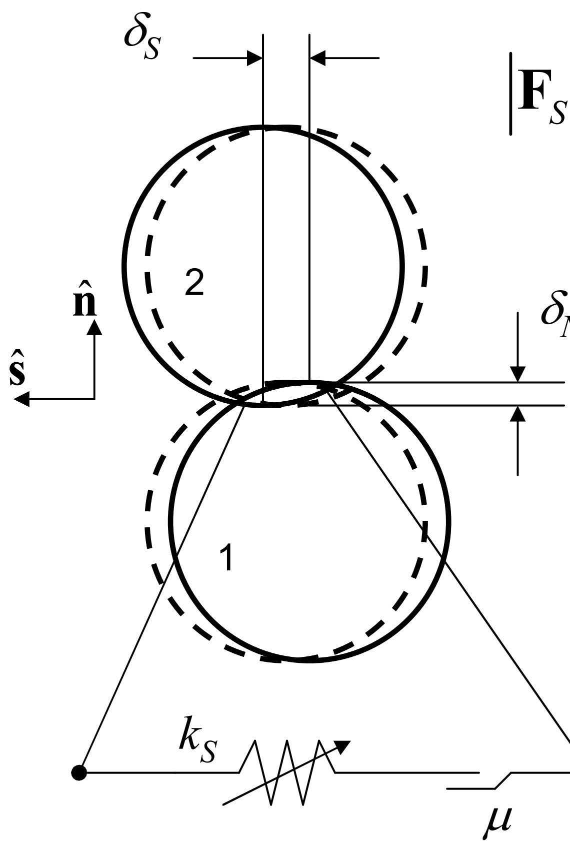
$$k_S = \frac{\sqrt{2RE}}{(2-\nu)(1+\nu)}\delta_N^{1/2}$$

  - Their tangential damping coefficient is set equal to the normal damping coefficient, which is given by

$$\nu_S = \nu_N = \alpha\sqrt{mk_N}\,\delta_N^{1/4} \qquad (k_N = k_{Hz})$$

    where α is an empirically determined constant
  - derived assuming a damped Hertzian normal spring force and constant normal coefficient of restitution

Incrementally Slipping Friction



$$\left|\mathbf{F}_{S,\text{on }1}\right| = \left|\mathbf{F}'_S\right| + k_S\left|\delta_S - \delta'_S\right| \qquad k_S = f k_S^0\left(1 - \frac{F_S - F_S^*}{\mu F_N - f F_S^*}\right)^{1/3}$$

$$f = \begin{cases} 1 & \dot{\delta}_S \text{ in initial direction} \\ -1 & \dot{\delta}_S \text{ in opposite direction} \end{cases}$$

- primed (‘) quantities indicate value at previous time step
- asterisked (*) quantities indicate value at last direction reversal
- If F_N changes during contact, then $F^*_{S,\text{new}} = F^*_{S,\text{old}} * (F_{N,\text{new}}/F_{N,\text{old}})$

Incrementally Slipping Friction…

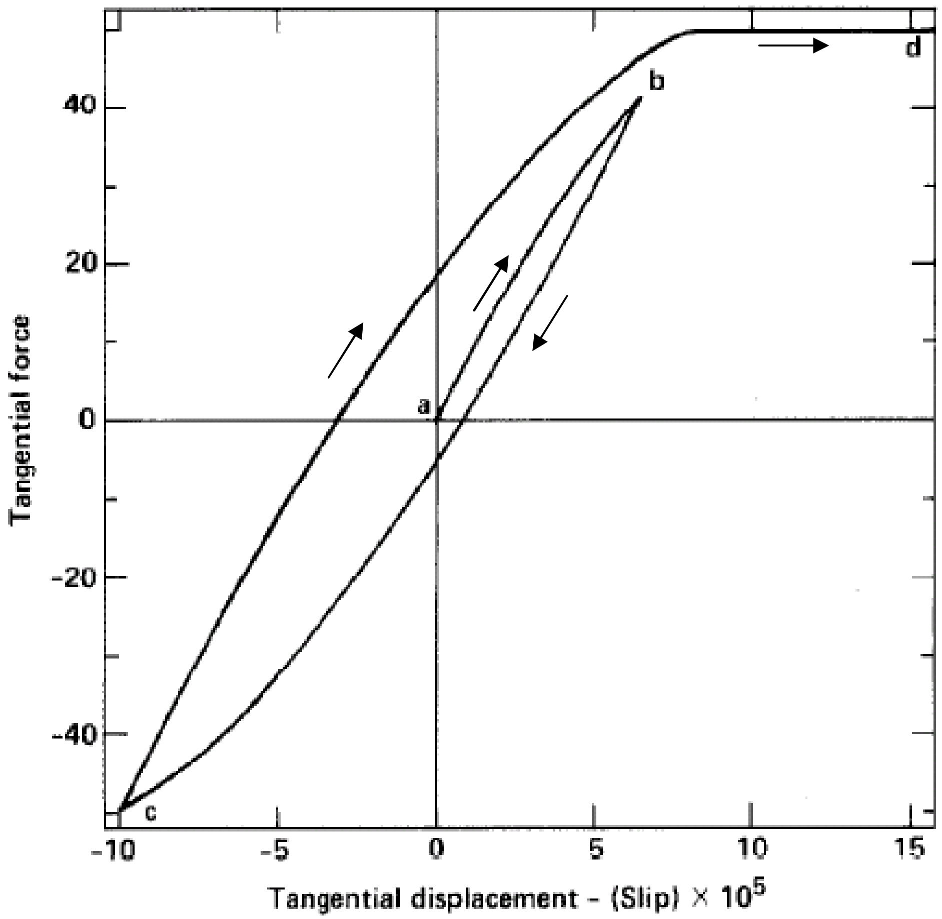


Fig. 3. Tangential force generated by incrementally slipping friction model with a constant normal force and ever increasing amplitude alternating tangential displacements (γ = 1/3 in Eq. 6).

a: initial loading
b: direction reversal
c: direction reversal
d: slipping

(figure assumes constant normal force)

From Walton and Braun (1986)

Incrementally Slipping Friction…

- Common, but not as ubiquitous as linear spring with sliding friction
- Most difficult model to implement of the ones presented so far – need to remember contact history
- Variable tangential stiffness
- Is possible to have velocity reversal
- Approximates Mindlin and Deresiewicz's (1953) solid mechanics model for tangential interactions
- Assumes that the normal force remains constant between time steps (justified for a gradually changing normal force)
- Matches experiments of Foerster *et al*. (1994) slightly better than Cundall-Strack model.
- Extension from 2-D to 3-D case is complicated by the need for a definition of "direction reversal".
  - more detailed notes on the implementation of this model are available in Walton (1993)
- Walton and Braun (1986) observed that the stress and velocity measurements in their 2D Couette flow simulations were not sensitive to the value of k_S^0/k_N over the range $2/3 < k_S^0/k_N < 1$.

Comparison with Experiments

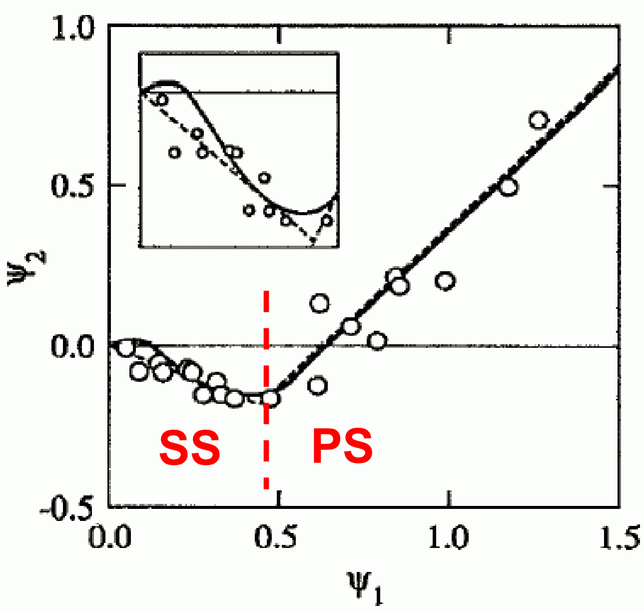


FIG. 4. Results for binary collisions of 3 mm glass spheres. The dashed line is a least-squares fit of the data through Eqs. (14) and (15). The solid line is the corresponding prediction of the model of Maw, Barber, and Fawcett.[7,10] The insert is an enlarged view of the region where sticking contacts occur.

From Foerster *et al*. (1994)

$$\psi_1 = -\left(\frac{\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{s}}}{\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{n}}}\right)$$ dimensionless pre-collision tangential speed

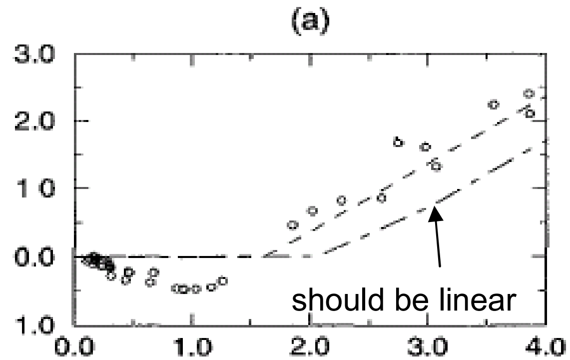
$$\psi_2 = -\left(\frac{\Delta\dot{\mathbf{x}}_c^+ \cdot \hat{\mathbf{s}}}{\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{n}}}\right)$$ dimensionless post-collision tangential speed

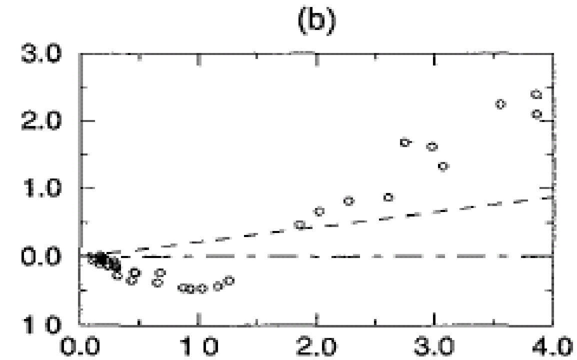
$$\psi_2^{SS} = -\varepsilon_S^{SS}\psi_1$$

$$\psi_2^{PS} = \psi_1 - \tfrac{7}{2}\mu\left(1+\varepsilon_N\right)\text{sign}\left(\psi_1\right)$$

Comparisons

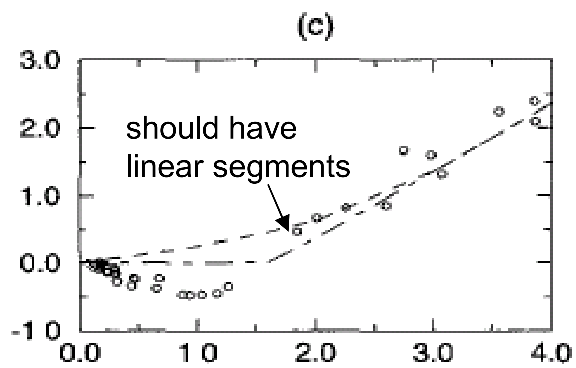
sliding friction
(μ = 0.15, 0.25)

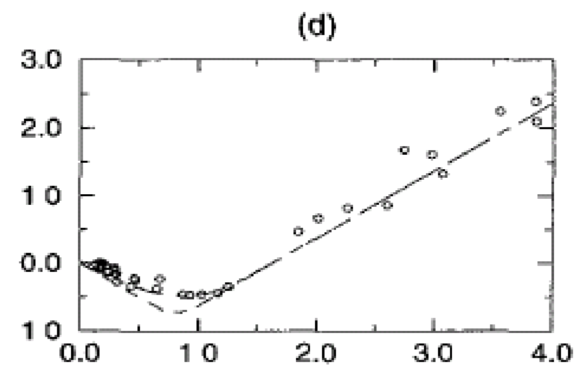




viscous damping
(ν_S = 2, 20 kg/s)

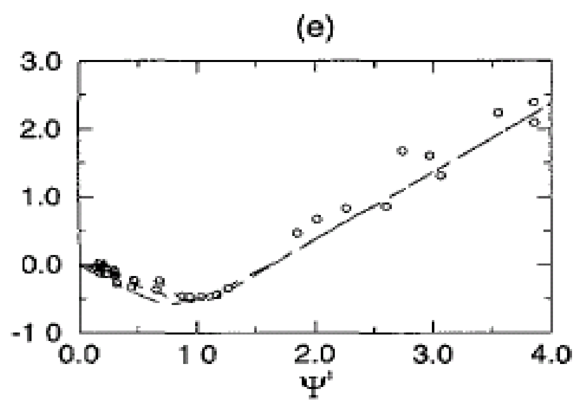
viscous damping w/ sliding friction
(μ = 0.25;
ν_S = 3, 20 kg/s)





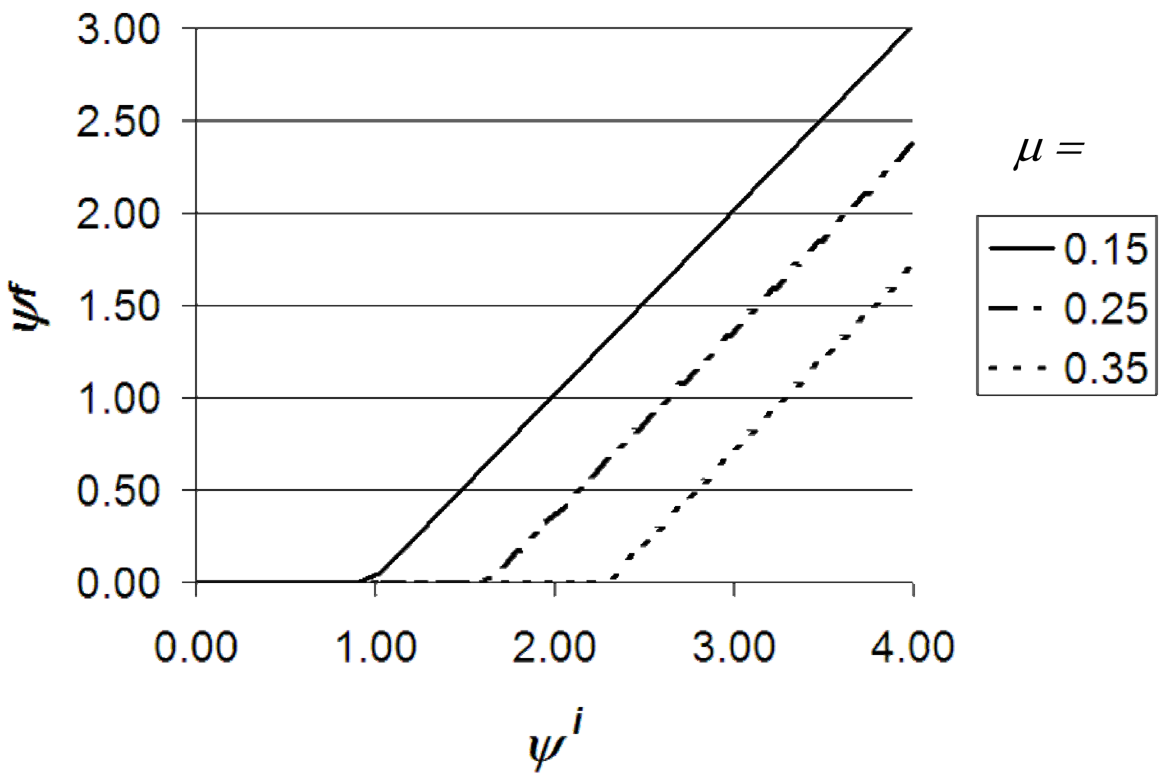
linear spring w/ sliding friction
(k_S/k_N = 2/7,1/5;
μ = 0.25)

incrementally slipping friction
(k^0_S/k_N = 2/3, 1/3;
μ = 0.25)



From Schäfer *et al.* (1996)

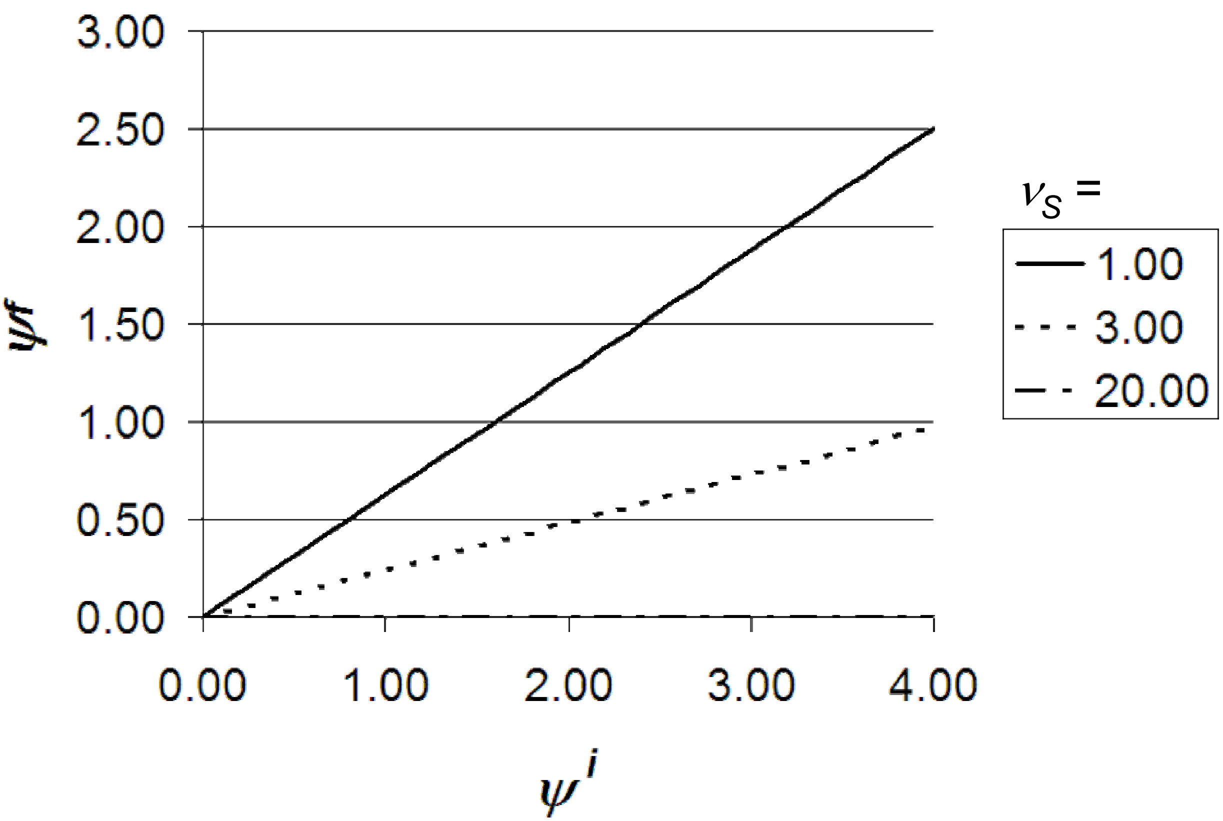
Comparisons: Sliding Friction



Two regimes are observed:

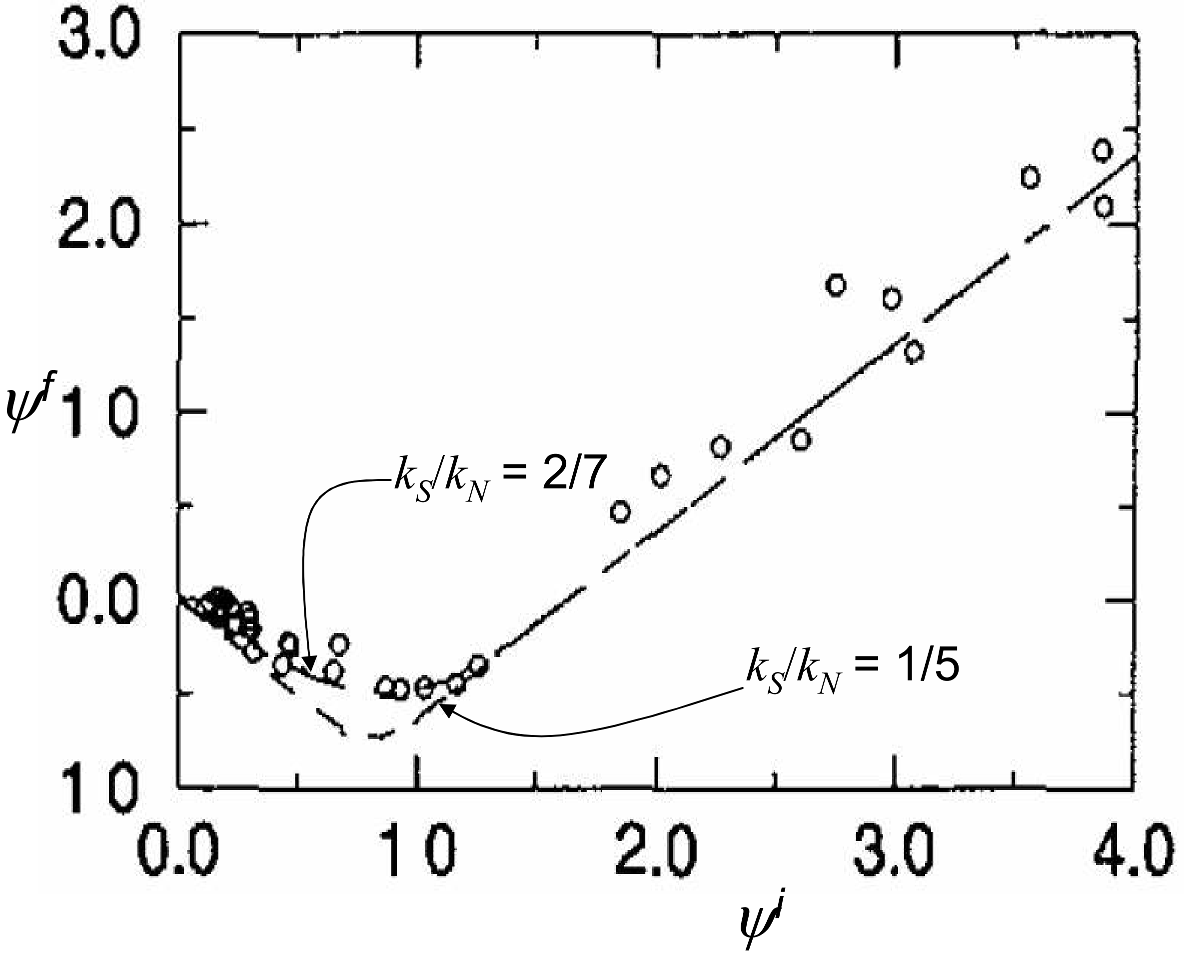
1. Stick regime for small ψ_i where particles roll against each other and tangential velocity is slowed to zero
2. Sliding regime for larger ψ_i where a finite, but slower tangential velocity results
3. Both regimes have constant slope, friction coefficient dictates transition between regimes

Comparisons: Viscous Damping



1. There is no stick regime, i.e. no rolling.
2. All curves have constant slope, *i.e.* model results in a constant tangential coefficient of restitution
3. Damping coefficient dictates slope

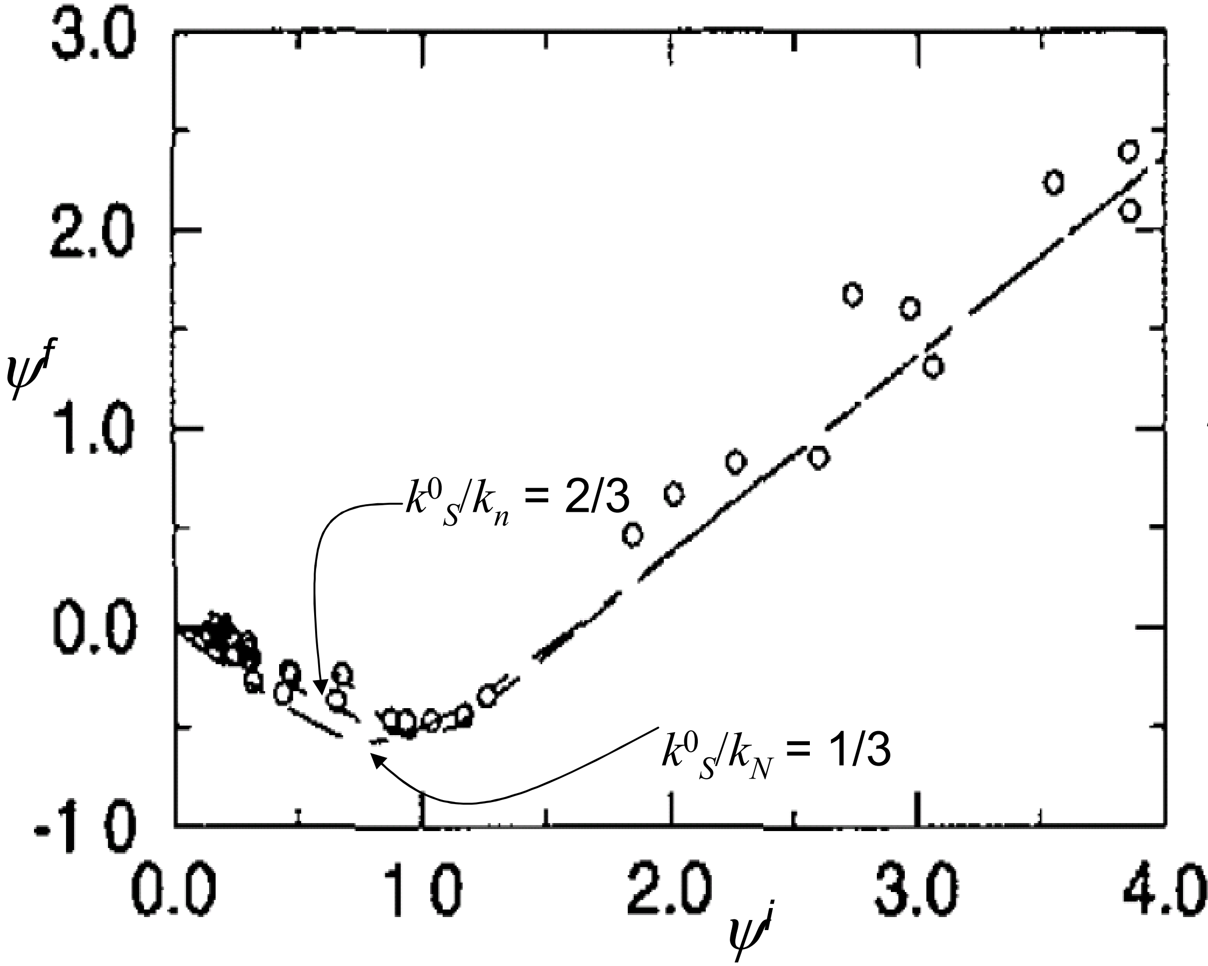
Comparisons: Linear Spring w/ Sliding Friction



1. All results are shown for μ = 0.25
2. Tangential stiffness allows for velocity reversal
3. Friction coefficient dictates ψ_i value where direction reversal occurs
4. Tangential stiffness / normal stiffness ratio dictates the magnitude of the direction reversal for low ψ_i impacts.

From Schäfer *et al.* (1996)

Comparisons: Incrementally Slipping Friction



From Schäfer *et al.* (1996)

1. Tangential stiffness allows for direction reversal
2. Very similar to Cundall-Strack model (with added complexity)

Comparisons…

- Little difference between linear spring with sliding and incrementally slipping models for binary collisions – both model experimental data well
  - pure sliding model also works well, but does not capture experimental observed velocity reversal
  - no investigation of model differences for densely packed, highly loaded systems
- The tangential force is typically a function of the normal force, so the choice of normal force model will influence the response of the tangential force model.
  - There has been essentially no investigation as to how the tangential contact model is influenced by the normal contact model

Comparisons…

- Sliding friction
  - Schäfer and Wolf (1995)
- Viscous damping
  - Gallas *et al*. (1992)
  - Melin (1994)
  - Luding *et al*. (1994)
  - Zhang and Campbell (1992) (but the force is placed at the centroid)
- Viscous damping with sliding friction
  - Kondic (1999)
  - Pöschel and Buchholtz (1993)
  - Pöschel and Herrmann (1995)
  - Thompson and Grest (1991)
  - Haff and Werner (1986)
- Linear damped spring with sliding friction
  - Cundall and Strack (1979)
  - Lee (1994) (dashpot + min(spring force, sliding friction force))
  - Lee and Herrmann (1993) (dashpot + min(spring force, sliding friction force))
  - Ristow and Herrmann (1994)
  - Taguchi (1992) (damped linear spring – no friction)
  - Tsuji *et al*. (1992)
- Variable spring stiffness with sliding friction
  - Sadd *et al.* (1993)
  - Lan and Rosato (1997)
  - Walton and Braun (1986)

Rolling Friction

- Due to asymmetric traction distribution over contact area

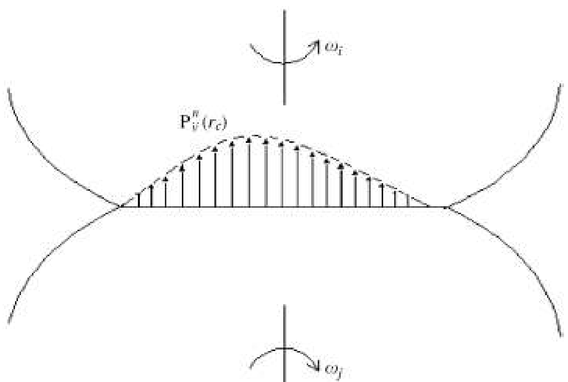


Fig. 1. Normal traction distribution exerted on particle i due to the collision with particle j.

Figure from Zhu and Yu (2003)

Kondic (1999)

$$\mathbf{F}_{\substack{\text{rolling,}\\ \text{on 1}}} = \mu_R \left|\mathbf{F}_N\right| \hat{\mathbf{v}}' \quad \text{where} \quad \hat{\mathbf{v}}' = \mathbf{v}_2 - \mathbf{v}_1$$

v is the center of mass translational velocity
$\mu_R \sim 10^{-3}$

Zhou *et al*. (2002)

$$\mathbf{T}_{\substack{\text{rolling,}\\ \text{on 1}}} = -\mu_R \left|\mathbf{F}_N\right| \hat{\boldsymbol{\omega}}$$

ω is the particle's rotational velocity
$0 \le \mu_R \le 0.2$ mm

Zhu and Yu (2003)

$$\mathbf{T}_{\substack{\text{rolling,}\\ \text{on 1}}} = \mu_R \min\left(\left|\mathbf{F}_N\right|, \left|\boldsymbol{\omega}'\right|\right) \hat{\boldsymbol{\omega}}'$$

$\omega' = \omega_2 - \omega_1$ (rotational velocity of 2 relative to 1 – note that both must be in the same FOR)
$\mu_R = 0.004d – 0.006d \approx 0.01\mu_{sliding}$ where d is the particle diameter

Note: μ_R should have different units depending on whether $\mathbf{F}_N$ or ω' is used, but there is no mention of this by the authors.

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