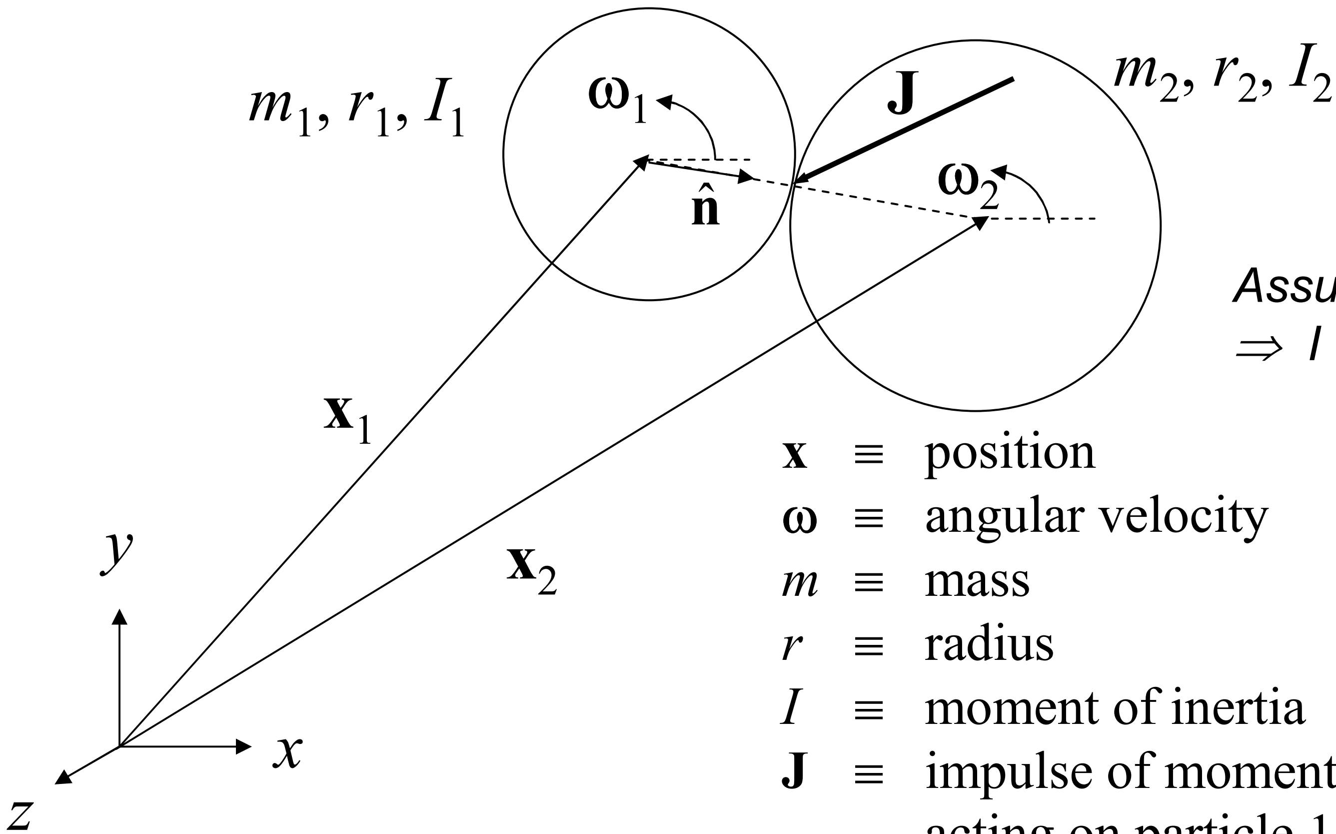
DEM Modeling: Lecture 02

Introduction to the Hard-Particle Algorithm Collision Model

Introduction

- Hard particle simulations are deterministic - given the same initial conditions the final results will be the same
  - the state of every particle in the system and all particle interactions are determined using physical laws
- The term “hard-particle” refers to the fact that particles are considered to be perfectly rigid
  - $\Rightarrow$ particles do not deform during a collision
  - $\Rightarrow$ collisions occur instantaneously
  - $\Rightarrow$ only binary (two particle) collisions occur
- Used most often for modeling dilute, energetic granular flows

Collision Dynamics



Assuming spheres
$\Rightarrow$ $I = {}^{2}/_{5}mr^2$

$\mathbf{x}$	$\equiv$	position
$\boldsymbol{\omega}$	$\equiv$	angular velocity
m	$\equiv$	mass
r	$\equiv$	radius
I	$\equiv$	moment of inertia
$\mathbf{J}$	$\equiv$	impulse of momentum acting on particle 1 due to collision with particle 2
$\hat{\mathbf{n}}$	$\equiv$	unit vector pointing from the center of particle 1 to the center of particle 2

$$\mathbf{J} = \int_{t_{\text{collision}}} \mathbf{F}\,dt$$

Collision Dynamics…

- Unit normal vector for the contact (assuming spheres)

$$\boxed{\hat{\mathbf{n}} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{\left|\mathbf{x}_2 - \mathbf{x}_1\right|}}$$

- Velocity of particle 2 relative to particle 1 at the point of contact

$$\dot{\mathbf{x}}_{1,C} = \dot{\mathbf{x}}_1 + \boldsymbol{\omega}_1 \times r_1\hat{\mathbf{n}}$$

$$\dot{\mathbf{x}}_{2,C} = \dot{\mathbf{x}}_2 + \boldsymbol{\omega}_2 \times \left(-r_2\hat{\mathbf{n}}\right)$$

$$\Delta\dot{\mathbf{x}}_C = \dot{\mathbf{x}}_{2,C} - \dot{\mathbf{x}}_{1,C} = \left(\dot{\mathbf{x}}_2 - \dot{\mathbf{x}}_1\right) - \left(\boldsymbol{\omega}_2 \times r_2\hat{\mathbf{n}} + \boldsymbol{\omega}_1 \times r_1\hat{\mathbf{n}}\right)$$

Collision Dynamics…

- Unit tangential vector for the contact
  - perpendicular to unit normal and points in the direction of the relative contact velocity just prior to the collision

$$\Delta\dot{\mathbf{x}}_C^- = \left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{s}}\right)\hat{\mathbf{s}}$$

$$\boxed{\hat{\mathbf{s}} = \frac{\Delta\dot{\mathbf{x}}_C^- - \left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left|\Delta\dot{\mathbf{x}}_C^- - \left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}\right|}}$$

- Normal and tangential components of the impact velocity

$$\dot{n} = \Delta\dot{\mathbf{x}}_C \cdot \hat{\mathbf{n}}$$

$$\dot{s} = \Delta\dot{\mathbf{x}}_C \cdot \hat{\mathbf{s}}$$

Collision Dynamics…

- Conservation of linear momentum for each particle

$$m_1\left(\dot{\mathbf{x}}_1^+ - \dot{\mathbf{x}}_1^-\right) = \mathbf{J}$$
$$m_2\left(\dot{\mathbf{x}}_2^+ - \dot{\mathbf{x}}_2^-\right) = -\mathbf{J}$$
$$\Rightarrow \boxed{\begin{aligned}\dot{\mathbf{x}}_1^+ &= \dot{\mathbf{x}}_1^- + \frac{\mathbf{J}}{m_1}\\ \dot{\mathbf{x}}_2^+ &= \dot{\mathbf{x}}_2^- - \frac{\mathbf{J}}{m_2}\end{aligned}}$$

- Conservation of angular momentum for each particle

$$I_1\left(\boldsymbol{\omega}_1^+ - \boldsymbol{\omega}_1^-\right) = r_1\hat{\mathbf{n}}\times\mathbf{J}$$
$$I_2\left(\boldsymbol{\omega}_2^+ - \boldsymbol{\omega}_2^-\right) = \left(-r_2\hat{\mathbf{n}}\right)\times\left(-\mathbf{J}\right)$$
$$\Rightarrow \boxed{\begin{aligned}\boldsymbol{\omega}_1^+ &= \boldsymbol{\omega}_1^- + \frac{r_1}{I_1}\left(\hat{\mathbf{n}}\times\mathbf{J}\right)\\ \boldsymbol{\omega}_2^+ &= \boldsymbol{\omega}_2^- + \frac{r_2}{I_2}\left(\hat{\mathbf{n}}\times\mathbf{J}\right)\end{aligned}}$$

Collision Dynamics…

- Consider the change in the relative contact speed after and before the collision

$$
\begin{aligned}
\Delta\dot{\mathbf{x}}_C^+ - \Delta\dot{\mathbf{x}}_C^- &= \left(\dot{\mathbf{x}}_2^+ - \dot{\mathbf{x}}_2^-\right) - \left(\dot{\mathbf{x}}_1^+ - \dot{\mathbf{x}}_1^-\right) - r_2\left(\boldsymbol{\omega}_2^+ - \boldsymbol{\omega}_2^-\right)\times\hat{\mathbf{n}} - r_1\left(\boldsymbol{\omega}_1^+ - \boldsymbol{\omega}_1^-\right)\times\hat{\mathbf{n}} \\
&= -\frac{\mathbf{J}}{m_2} - \frac{\mathbf{J}}{m_1} - \frac{r_2^2}{I_2}\left(\hat{\mathbf{n}}\times\mathbf{J}\right)\times\hat{\mathbf{n}} - \frac{r_1^2}{I_1}\left(\hat{\mathbf{n}}\times\mathbf{J}\right)\times\hat{\mathbf{n}} \\
&= -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{J} - \left(\frac{r_1^2}{I_1} + \frac{r_2^2}{I_2}\right)\left[\left(\hat{\mathbf{n}}\times\mathbf{J}\right)\times\hat{\mathbf{n}}\right]
\end{aligned}
$$

- Make use of a vector identity

$$
\left(\hat{\mathbf{n}}\times\mathbf{J}\right)\times\hat{\mathbf{n}} = \underbrace{\left(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\right)}_{=1}\mathbf{J} - \left(\mathbf{J}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}} = \mathbf{J} - \left(\mathbf{J}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}
$$

- Also use an effective mass

$$
1/m' = 1/m_1 + 1/m_2
$$

Collision Dynamics…

- Simplify

$$\Delta\dot{\mathbf{x}}_C^{+} - \Delta\dot{\mathbf{x}}_C^{-} = -\frac{1}{m'}\mathbf{J} - \left(\frac{r_1^2}{I_1} + \frac{r_2^2}{I_2}\right)\left[\mathbf{J} - \left(\mathbf{J}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}\right]$$

$$= -\left(\frac{1}{m'} + \frac{r_1^2}{I_1} + \frac{r_2^2}{I_2}\right)\mathbf{J} + \left(\frac{r_1^2}{I_1} + \frac{r_2^2}{I_2}\right)\left(\mathbf{J}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$$

- Write the normal and tangential components of the change in the relative impact speed

$$\left(\Delta\dot{\mathbf{x}}_C^{+} - \Delta\dot{\mathbf{x}}_C^{-}\right)\cdot\hat{\mathbf{n}} = -\left(\frac{1}{m'} + \frac{r_1^2}{I_1} + \frac{r_2^2}{I_2}\right)\left(\mathbf{J}\cdot\hat{\mathbf{n}}\right) + \left(\frac{r_1^2}{I_1} + \frac{r_2^2}{I_2}\right)\left(\mathbf{J}\cdot\hat{\mathbf{n}}\right) = -\frac{1}{m'}\left(\mathbf{J}\cdot\hat{\mathbf{n}}\right)$$

$$\left(\Delta\dot{\mathbf{x}}_C^{+} - \Delta\dot{\mathbf{x}}_C^{-}\right)\cdot\hat{\mathbf{s}} = -\left(\frac{1}{m'} + \frac{r_1^2}{I_1} + \frac{r_2^2}{I_2}\right)\left(\mathbf{J}\cdot\hat{\mathbf{s}}\right)$$

Collision Dynamics…

- Now consider the normal component of the impulse

$$\mathbf{J}\cdot\hat{\mathbf{n}} = -m'\left(\Delta\dot{\mathbf{x}}_C^+ - \Delta\dot{\mathbf{x}}_C^-\right)\cdot\hat{\mathbf{n}} = -m'\left[\left(\Delta\dot{\mathbf{x}}_C^+\cdot\hat{\mathbf{n}}\right) - \left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{n}}\right)\right]$$

- Make use of the definition of the normal coefficient of restitution, ε_N

$$\varepsilon_N \equiv -\left(\frac{\Delta\dot{\mathbf{x}}_C^+\cdot\hat{\mathbf{n}}}{\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{n}}}\right) \Rightarrow \left(\Delta\dot{\mathbf{x}}_C^+\cdot\hat{\mathbf{n}}\right) = -\varepsilon_N\left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{n}}\right)$$

(Note: $0 \le \varepsilon_N \le 1$ with $\varepsilon_N = 0$ being completely inelastic and $\varepsilon_N = 1$ being perfectly elastic.

$$\boxed{\mathbf{J}\cdot\hat{\mathbf{n}} = m'\left(1+\varepsilon_N\right)\left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{n}}\right)}$$

Collision Dynamics…

- Consider the tangential component of the impulse

$$\mathbf{J}\cdot\hat{\mathbf{s}} = -\left(\frac{1}{m'}+\frac{r_1^2}{I_1}+\frac{r_2^2}{I_2}\right)^{-1}\left(\Delta\dot{\mathbf{x}}_C^+ - \Delta\dot{\mathbf{x}}_C^-\right)\cdot\hat{\mathbf{s}} = -\left(\frac{1}{m'}+\frac{r_1^2}{I_1}+\frac{r_2^2}{I_2}\right)^{-1}\left[\left(\Delta\dot{\mathbf{x}}_C^+\cdot\hat{\mathbf{s}}\right)-\left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}\right)\right]$$

- Make use of the definition of the tangential coefficient of restitution, ε_S

$$\varepsilon_S \equiv -\left(\frac{\Delta\dot{\mathbf{x}}_C^+\cdot\hat{\mathbf{s}}}{\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}}\right) \Rightarrow \left(\Delta\dot{\mathbf{x}}_C^+\cdot\hat{\mathbf{s}}\right) = -\varepsilon_S\left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}\right)$$

(Note: -1 ≤ ε_S ≤ 1 with ε_S = -1 being frictionless, ε_S = 0 resulting in no-slip and ε_S = 1 being perfectly elastic.

$$\mathbf{J}\cdot\hat{\mathbf{s}} = \left(\frac{1}{m'}+\frac{r_1^2}{I_1}+\frac{r_2^2}{I_2}\right)^{-1}\left(1+\varepsilon_S\right)\left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}\right)$$

Collision Dynamics…

- Re-write the collision impulse

$$\mathbf{J} = \left(\mathbf{J}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \left(\mathbf{J}\cdot\hat{\mathbf{s}}\right)\hat{\mathbf{s}}$$

$$= m'\left(1+\varepsilon_N\right)\left(\Delta\dot{\mathbf{x}}_C^- \cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \left(\frac{1}{m'}+\frac{r_1^2}{I_1}+\frac{r_2^2}{I_2}\right)^{-1}\left(1+\varepsilon_S\right)\left(\Delta\dot{\mathbf{x}}_C^- \cdot\hat{\mathbf{s}}\right)\hat{\mathbf{s}}$$

- Note that since we're assuming spheres, $I = {}^2/_5 mr^2$

$$\boxed{\mathbf{J} = m'\left(1+\varepsilon_N\right)\left(\Delta\dot{\mathbf{x}}_C^- \cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \tfrac{2}{7}m'\left(1+\varepsilon_S\right)\left(\Delta\dot{\mathbf{x}}_C^- \cdot\hat{\mathbf{s}}\right)\hat{\mathbf{s}}}$$

Collision Model Parameters

normal coefficient of restitution, ε_N

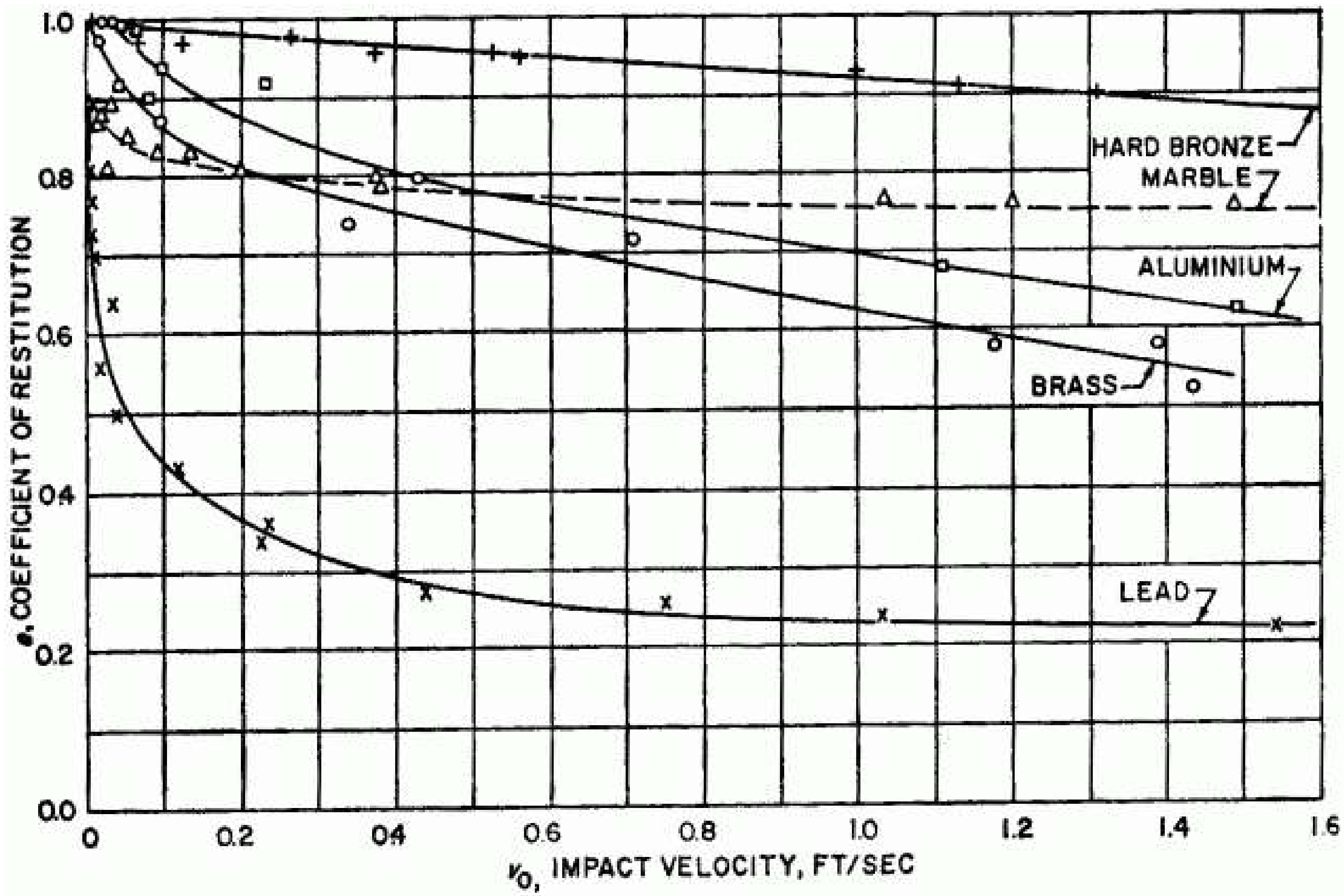
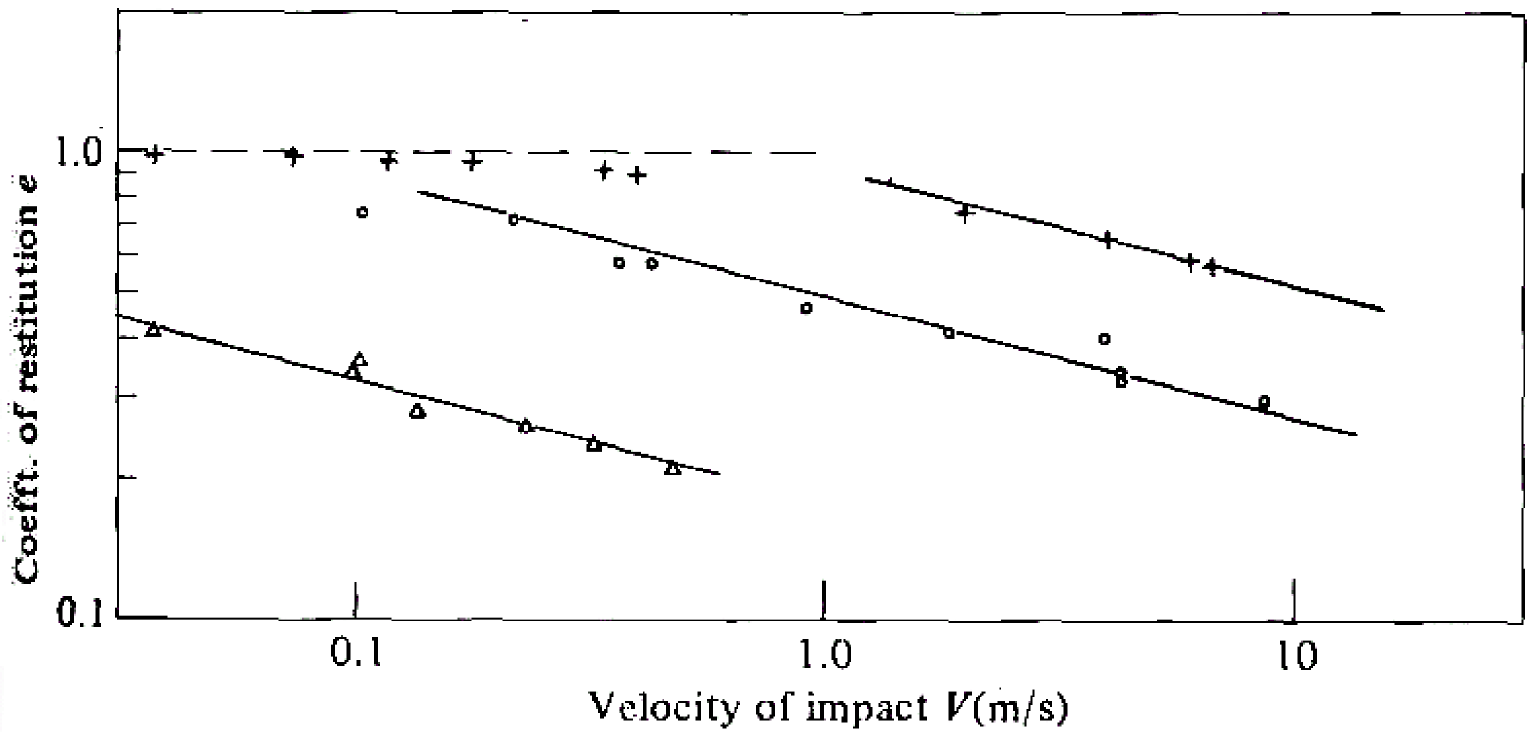


Figure 165. Coefficient of Restitution as a Function of Impact Velocity for Spheres of the Same Size and Material [302, 330]

Goldsmith, W., 1960, *Impact: The Theory and Physical Behaviour of Colliding Solids*, Dover.

Contact Model Parameters…

Fig. 11.11. Measurements of the coefficient of restitution of a steel ball on blocks of various materials (from Goldsmith, 1960). Cross – hard bronze; circle – brass; triangle – lead. Lines of slope $-\frac{1}{4}$.



Johnson, K.L., 1985, *Contact Mechanics*, Cambridge University Press.

Contact Model Parameters…

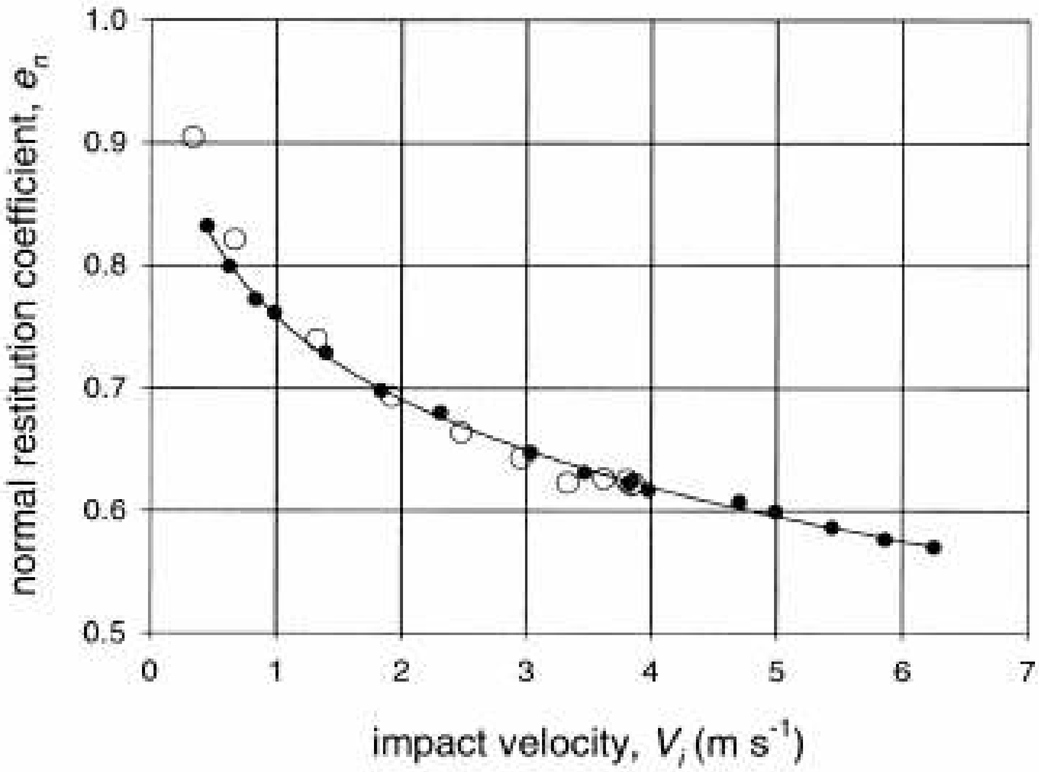


Fig. 12. Filled circles are measurements of restitution coefficient for normal impacts over a range of velocities. The line is Tabor's model (Eq. 10) fitted to these data. The open circles are e_n values from oblique impact experiments plotted against the normal velocity component. Aluminium alloy anvil.

normal and oblique impacts give similar normal coefficients of restitution

Gorham, D.A. and Kharaz, A.H., 2000, "The measurement of particle rebound characteristics," *Powder Technology*, Vol. 112, pp. 193 – 202.

Contact Model Parameters…

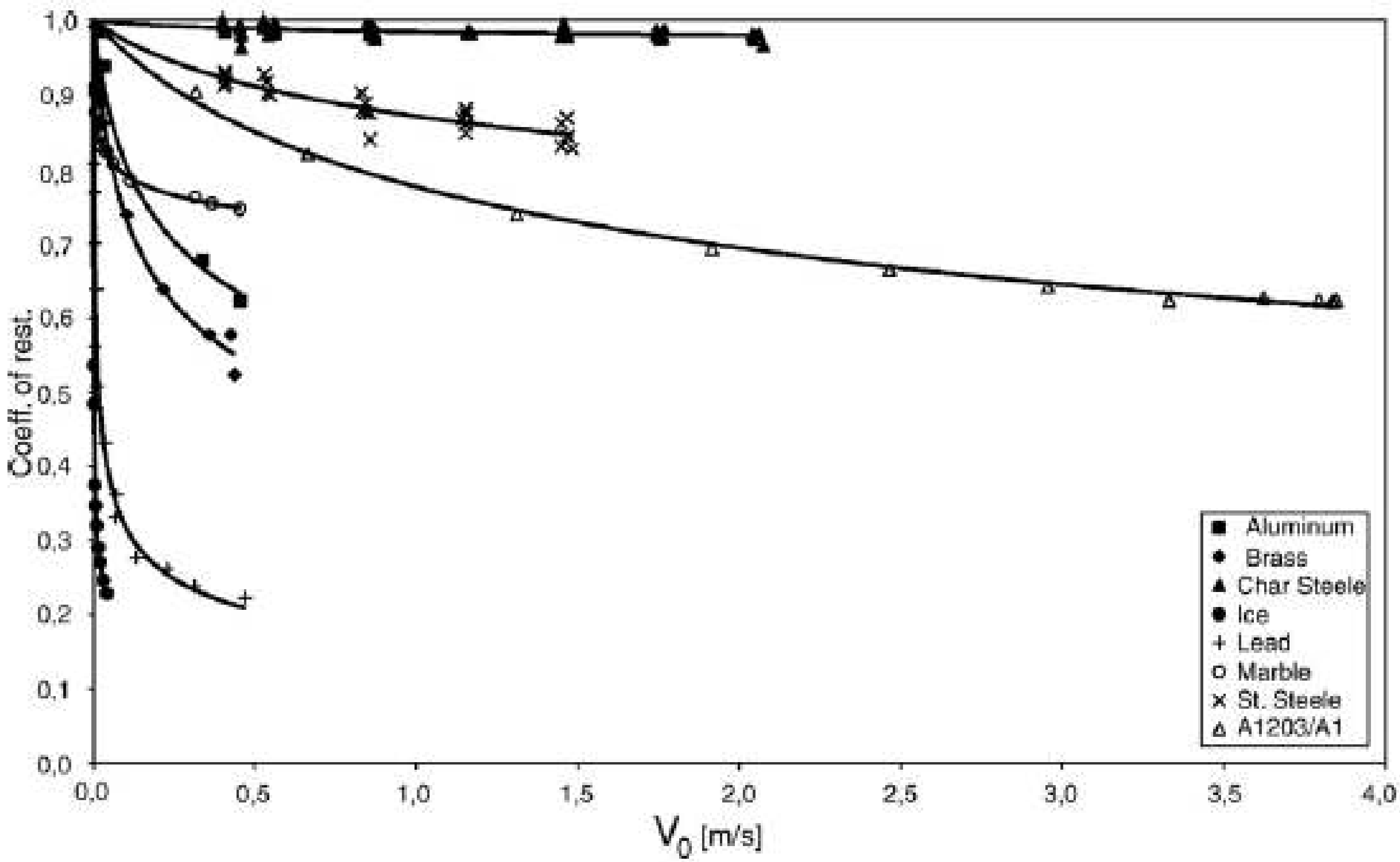


Fig. 3. Coefficient of restitution as a function of the initial normal velocity v_0 [23,28–30].

Kruggel-Emden, H., Simsek, E., Rickelt, S., Wirtz, S., and Scherer, V., 2007, "Review and extension of normal force models," *Powder Technology*, Vol. 171, pp. 157 – 173.

Contact Model Parameters…

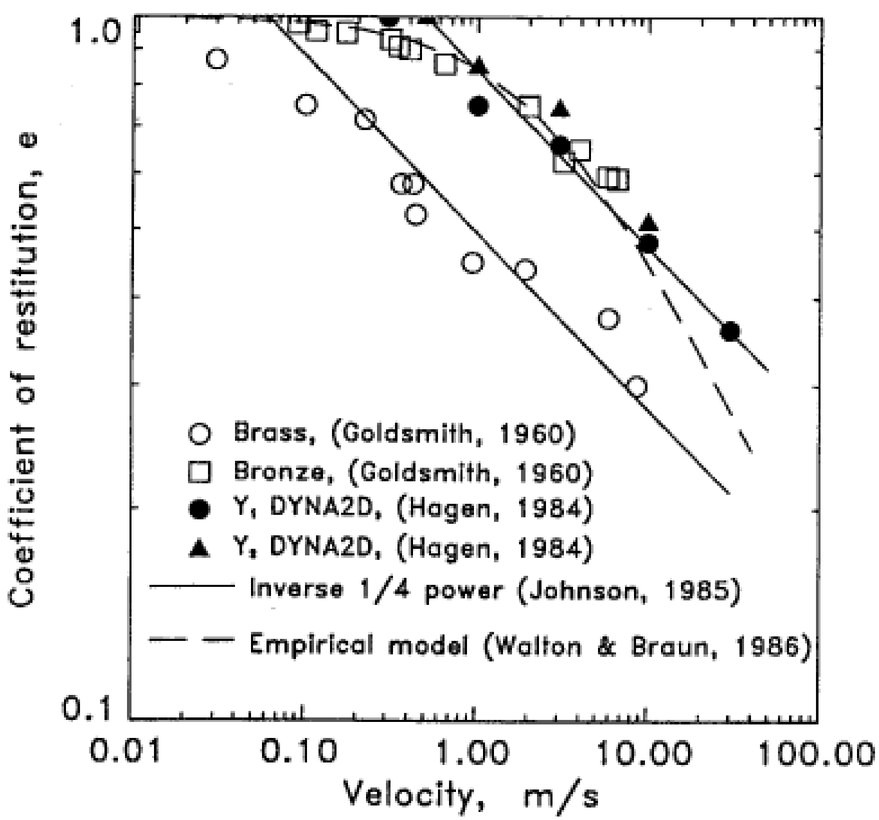


FIGURE 25-3 *Variation of coefficient of restitution with impact velocity. Experimental results (open symbols) are for metal spheres (Goldsmith, 1960). Finite element calculations for elastic perfectly-plastic material model are shown as filled circles (yield strength set so plastic deformation starts at impact velocities exceeding 0.3m/s) and pluses (yield set to allow plastic deformation only above 0.5m/s). Dashed line is representative curve for empirical formula of Walton and Braun (1986). Solid lines are inverse 1/4 power of velocity, as predicted by fully plastic theory (Johnson, 1986).*

Walton, O.R., 1993, "Numerical simulation of inelastic, frictional particle-particle interactions," in *Particulate Two-Phase Flow*, Chap. 25, M.C.Roco ed., Butterworth-Heinemann.

Collision Model Parameters…

- $(1 - \varepsilon_N) \sim V_{impact}^{1/5}$ as $\varepsilon_N \to 1$ in the visco-elastic regime
  - Kuwabara and Kono (1987)
  - V_{impact} < ~1 m/s
- $\varepsilon_N \sim V_{impact}^{-1/4}$ in the elasto-plastic regime
  - Johnson (1985)
  - V_{impact} > ~1 m/s

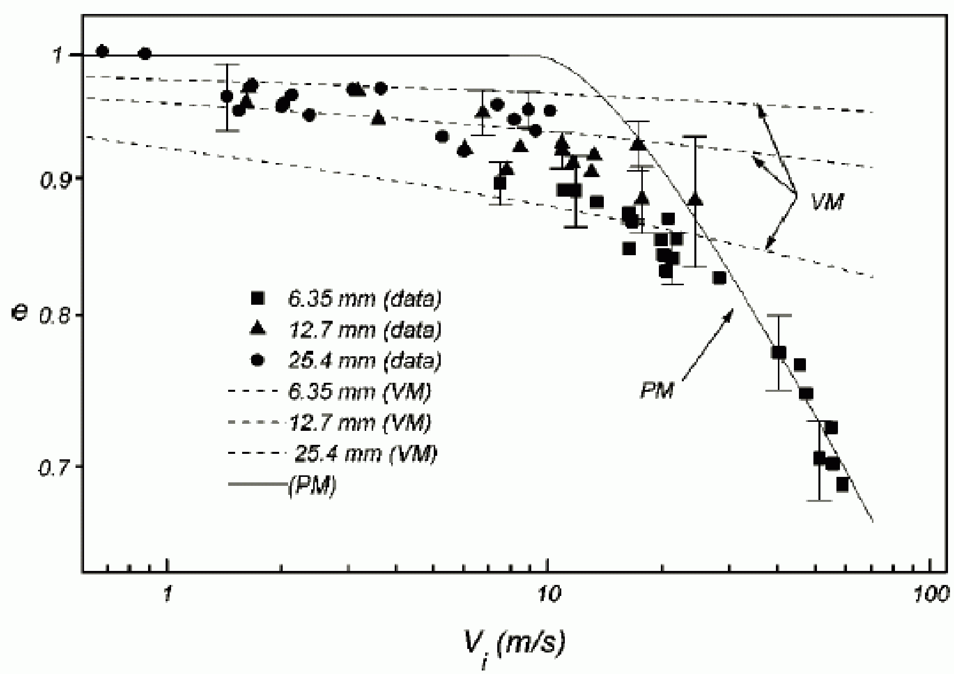


FIG. 7. Coefficient of restitution vs normal impact velocity for nylon spheres for different diameters on a log-log scale. The diameter of the spheres is shown in the figure. VM denotes the viscoelastic model. PM denotes plastic model.

Labous, L., Rosato, A.D., and Dave, R.N., 1997, "Measurements of collisional properties of spheres using high-speed video analysis," *Physical Review E*, Vol. 56, pp. 5717-5725.

Elastic-Perfectly Plastic Collisions

- Plastic yield starts to occur at

$$\boxed{V_{\text{yield}}^2 \approx 107 \frac{R'^3 Y^5}{m' E'^4}}$$

  - *e.g.* two 3 mm diameter stainless steel (grade 316) spheres (ρ = 8030 kg/m^3, E = 193 GPa, n = 0.35, Y = 310 MPa) $\Rightarrow V_{\text{yield}} \approx$ 4 mm/s
  - plastic yield is common

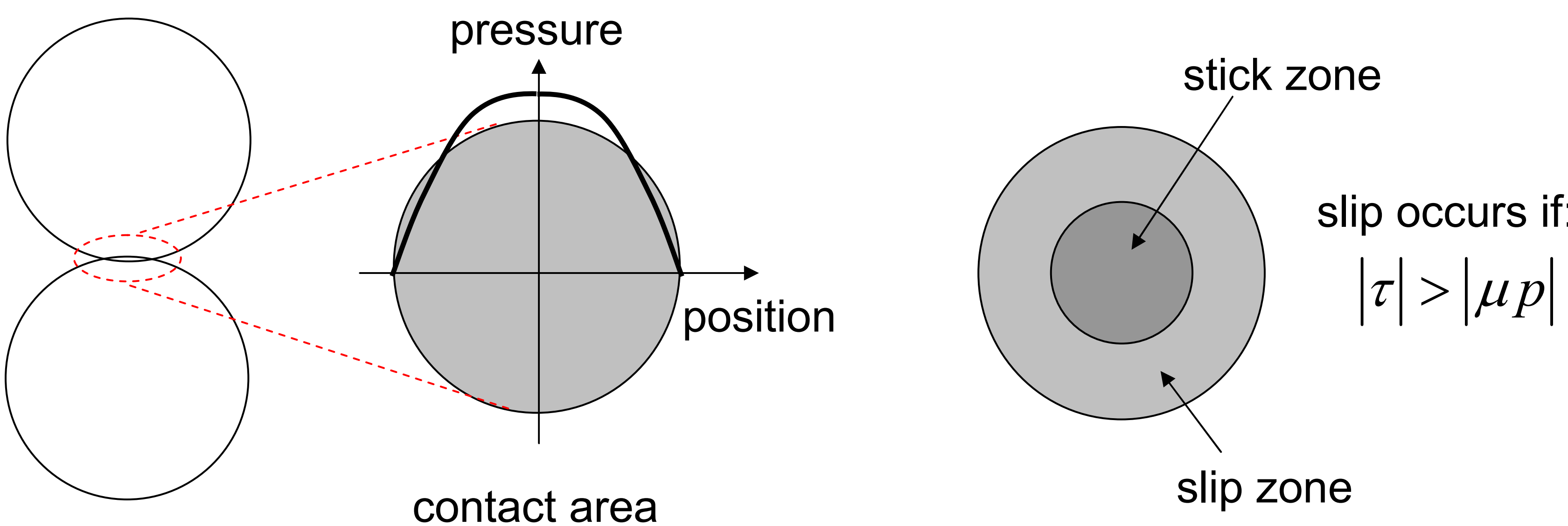
- Assuming fully plastic loading and elastic unloading:

$$\boxed{\varepsilon_N \approx \frac{3}{8}\left(\frac{Y}{E'}\right)^{1/2}\left(\frac{\frac{1}{2} m' V_{\text{impact}}^2}{YR'^3}\right)^{-1/8}}$$

Johnson, K.L., 1985, *Contact Mechanics*, Cambridge University Press.

Stick-Slip Contacts

- In a real contact, the contact area has regions that “stick” toward the center of the contact area and regions that “slip” at the edges



- For sufficiently small pressures, the entire contact area may slip

Stick-Slip Contacts…

- Assume that the contact has two tangential coefficients of restitution
  - a constant tangential coefficient of restitution in the stick-slip (SS) regime given by ε_S^{SS}
  - a variable tangential coefficient of restitution in the pure slip (PS) regime denoted by ε_S^{PS}

Stick-Slip Contacts…

- In the stick-slip regime:

$$\left(\Delta\dot{\mathbf{x}}_c^+ \cdot \hat{\mathbf{s}}\right) = -\varepsilon_S \left(\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{s}}\right) \Rightarrow \frac{\left(\Delta\dot{\mathbf{x}}_c^+ \cdot \hat{\mathbf{s}}\right)}{\left(\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{n}}\right)} = -\varepsilon_S \frac{\left(\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{s}}\right)}{\left(\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{n}}\right)} \Rightarrow \psi_2 = -\varepsilon_S \psi_1$$

$$\boxed{\therefore \psi_2^{SS} = -\varepsilon_S^{SS} \psi_1}$$

$$\boxed{\left(\mathbf{J} \cdot \hat{\mathbf{s}}\right)^{SS} = \tfrac{2}{7} m' \left(1 + \varepsilon_S^{SS}\right)\left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{s}}\right)}$$

Stick-Slip Contacts…

- In the pure-slip regime:

$$\left(\mathbf{J}\cdot\hat{\mathbf{s}}\right)^{PS}=\tfrac{2}{7}m'\left(1+\varepsilon_S^{PS}\right)\left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}\right)=\operatorname{sign}\left(\Delta\dot{\mathbf{x}}_c^-\cdot\hat{\mathbf{s}}\right)\mu\left|\mathbf{J}\cdot\hat{\mathbf{n}}\right|$$

  - the magnitude of the tangential momentum impulse is limited by sliding friction

$$\left(\mathbf{J}\cdot\hat{\mathbf{s}}\right)^{PS}\leq\left(\mathbf{J}\cdot\hat{\mathbf{s}}\right)^{SS}\Rightarrow\operatorname{sign}\left(\Delta\dot{\mathbf{x}}_c^-\cdot\hat{\mathbf{s}}\right)\mu\left|\mathbf{J}\cdot\hat{\mathbf{n}}\right|\leq\tfrac{2}{7}m'\left(1+\varepsilon_S^{SS}\right)\left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}\right)$$

$$\mu\leq\tfrac{2}{7}m'\left(1+\varepsilon_S^{SS}\right)\left|\frac{\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}}{\mathbf{J}\cdot\hat{\mathbf{n}}}\right|=\tfrac{2}{7}m'\left(1+\varepsilon_S^{SS}\right)\left|\frac{\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}}{m'\left(1+\varepsilon_N\right)\left(\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{n}}\right)}\right|$$

slipping occurs if: $\left|\dfrac{\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{s}}}{\Delta\dot{\mathbf{x}}_C^-\cdot\hat{\mathbf{n}}}\right|=\left|\psi_1\right|\geq\mu\dfrac{7}{2}\left(\dfrac{1+\varepsilon_N}{1+\varepsilon_S^{SS}}\right)$

Stick-Slip Contacts…

- In the pure-slip regime…

$$\left(\mathbf{J}\cdot\hat{\mathbf{s}}\right)^{PS} = \tfrac{2}{7}m'\left(1+\varepsilon_S^{PS}\right)\left(\Delta\dot{\mathbf{x}}_C^{-}\cdot\hat{\mathbf{s}}\right) = \operatorname{sign}\left(\Delta\dot{\mathbf{x}}_c^{-}\cdot\hat{\mathbf{s}}\right)\mu\left|\mathbf{J}\cdot\hat{\mathbf{n}}\right|$$

$$\tfrac{2}{7}m'\left(1+\varepsilon_S^{PS}\right)\left(\Delta\dot{\mathbf{x}}_C^{-}\cdot\hat{\mathbf{s}}\right) = \operatorname{sign}\left(\Delta\dot{\mathbf{x}}_c^{-}\cdot\hat{\mathbf{s}}\right)\mu m'\left(1+\varepsilon_N\right)\left|\Delta\dot{\mathbf{x}}_C^{-}\cdot\hat{\mathbf{n}}\right|$$

$$\boxed{\varepsilon_S^{PS} = \frac{7}{2}\mu\left(1+\varepsilon_N\right)\left|\frac{\Delta\dot{\mathbf{x}}_C^{-}\cdot\hat{\mathbf{n}}}{\Delta\dot{\mathbf{x}}_C^{-}\cdot\hat{\mathbf{s}}}\right|-1}$$

$$\psi_2^{PS} = -\varepsilon_S^{PS}\psi_1 \Rightarrow \psi_2^{PS} = -\left[\frac{7}{2}\mu\left(1+\varepsilon_N\right)\left|\frac{\Delta\dot{\mathbf{x}}_C^{-}\cdot\hat{\mathbf{n}}}{\Delta\dot{\mathbf{x}}_C^{-}\cdot\hat{\mathbf{s}}}\right|-1\right]\psi_1$$

$$\boxed{\psi_2^{PS} = \psi_1 - \tfrac{7}{2}\mu\left(1+\varepsilon_N\right)\operatorname{sign}\left(\psi_1\right)}$$

Collision Model Parameters…

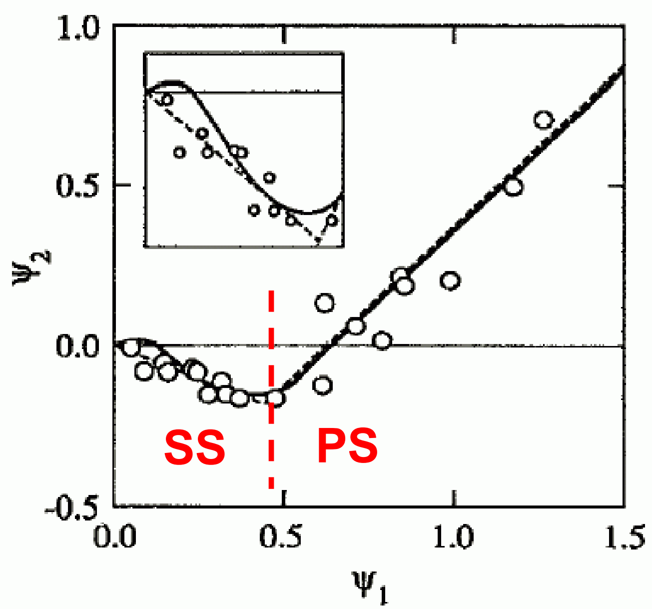


FIG. 4. Results for binary collisions of 3 mm glass spheres. The dashed line is a least-squares fit of the data through Eqs. (14) and (15). The solid line is the corresponding prediction of the model of Maw, Barber, and Fawcett.[7,10] The insert is an enlarged view of the region where sticking contacts occur.

$$\psi_1 = -\left(\frac{\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{s}}}{\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{n}}}\right)$$

dimensionless pre-collision tangential speed

$$\psi_2 = -\left(\frac{\Delta\dot{\mathbf{x}}_c^+ \cdot \hat{\mathbf{s}}}{\Delta\dot{\mathbf{x}}_c^- \cdot \hat{\mathbf{n}}}\right)$$

dimensionless post-collision tangential speed

$$\psi_2^{SS} = -\varepsilon_S^{SS}\psi_1$$

$$\psi_2^{PS} = \psi_1 - \tfrac{7}{2}\mu\left(1+\varepsilon_N\right)\mathrm{sign}\left(\psi_1\right)$$

Foerster, S.F., Louge, M.Y., Chang, H., and Allia, K., 1994, "Measurements of the collision properties of small spheres," *Physics of Fluids*, Vol. 6, pp. 1108-1115.

Collision Model Parameters…

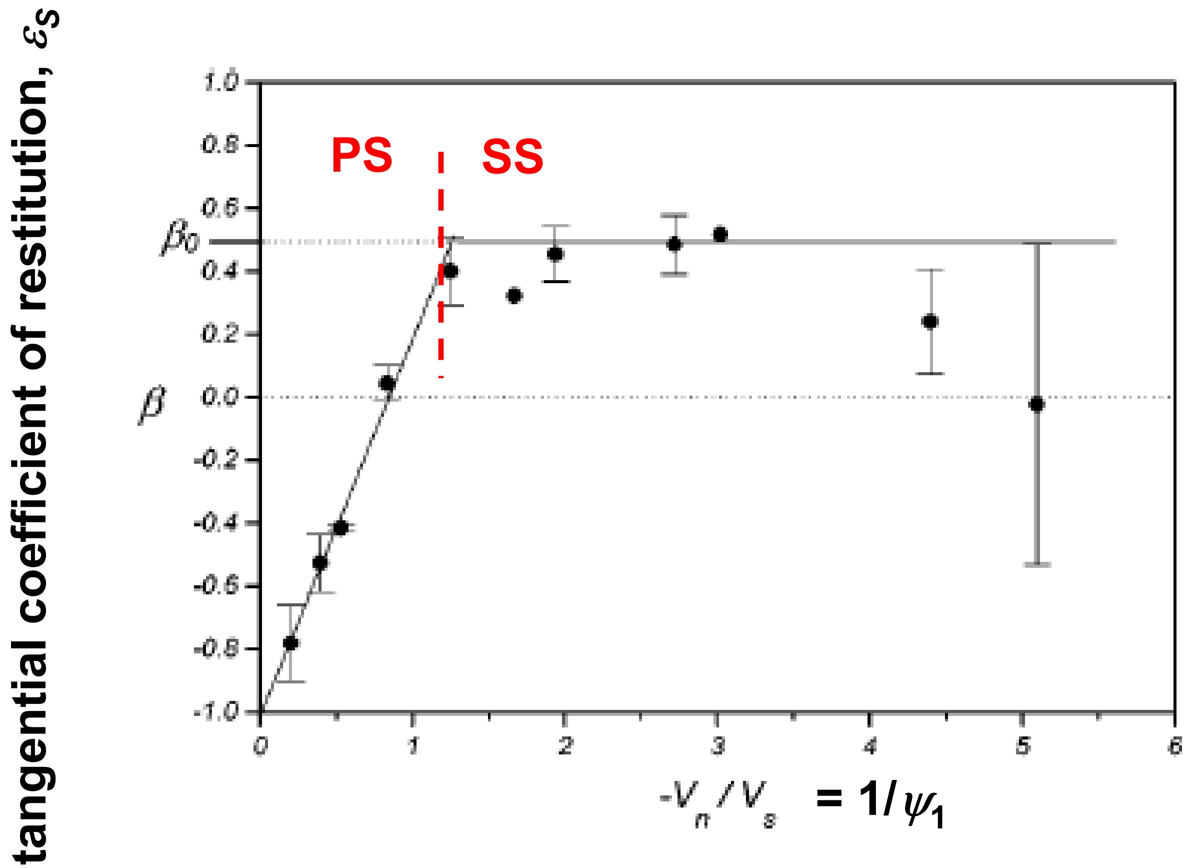
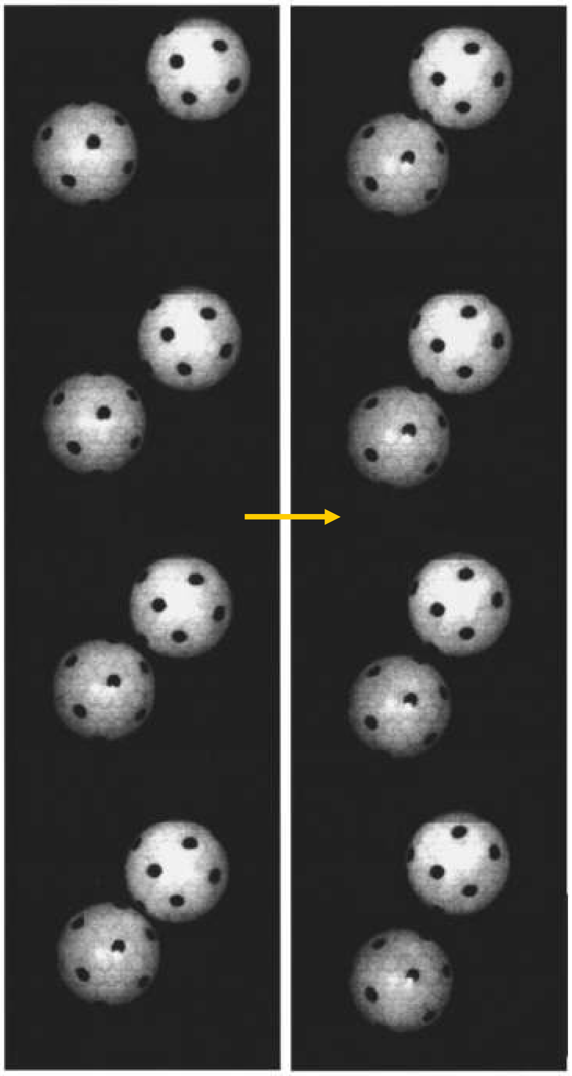


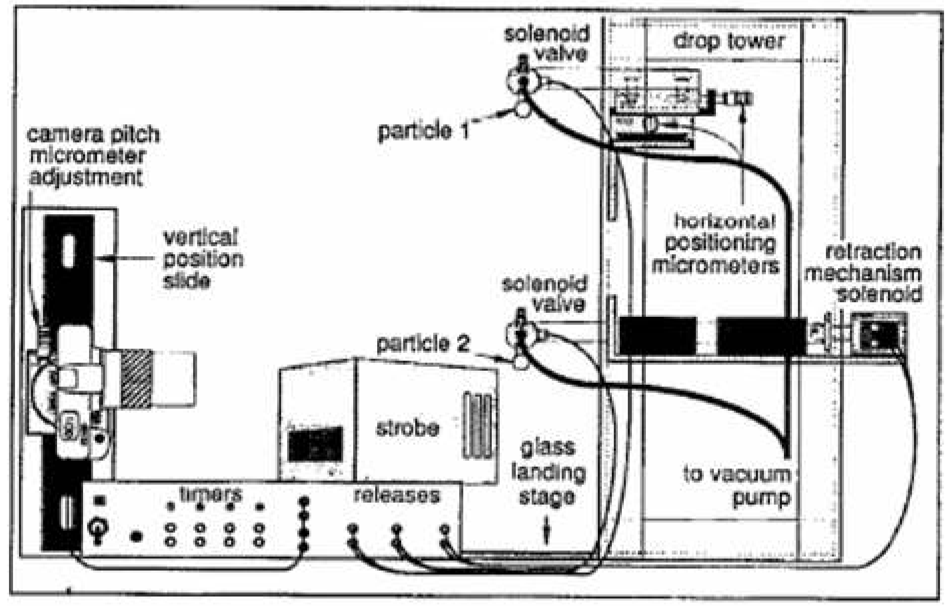
FIG. 6. Coefficient of tangential restitution β vs the cotangent of the angle of incidence.

Labous, L., Rosato, A.D., and Dave, R.N., 1997, "Measurements of collisional properties of spheres using high-speed video analysis," *Physical Review E*, Vol. 56, pp. 5717-5725.

Measurements Methods



Labous *et al*. (1997)



Foerster *et al*. (1994)

- High speed photography of marked, colliding particles.
- Typically use “large” (> ~1 mm) particles for easy visualization.

Summary

- Three parameter hard particle collision model

$$\frac{1}{m'} = \frac{1}{m_1} + \frac{1}{m_2} \qquad \hat{\mathbf{n}} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{\left|\mathbf{x}_2 - \mathbf{x}_1\right|}$$

$$\dot{\mathbf{x}}_{1,C} = \dot{\mathbf{x}}_1 + \boldsymbol{\omega}_1 \times r_1\hat{\mathbf{n}} \qquad \dot{\mathbf{x}}_{2,C} = \dot{\mathbf{x}}_2 + \boldsymbol{\omega}_2 \times \left(-r_2\hat{\mathbf{n}}\right) \qquad \Delta\dot{\mathbf{x}}_C = \dot{\mathbf{x}}_{2,C} - \dot{\mathbf{x}}_{1,C}$$

$$\hat{\mathbf{s}} = \frac{\Delta\dot{\mathbf{x}}_C^- - \left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left|\Delta\dot{\mathbf{x}}_C^- - \left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}\right|}$$

$$\varepsilon_S^{PS} = \frac{7}{2}\mu\left(1+\varepsilon_N\right)\left|\frac{\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{n}}}{\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{s}}}\right| - 1 \qquad \varepsilon_S = \min\left(\varepsilon_S^{SS}, \varepsilon_S^{PS}\right)$$

$$\mathbf{J} = m'\left(1+\varepsilon_N\right)\left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}} + \tfrac{2}{7}m'\left(1+\varepsilon_S\right)\left(\Delta\dot{\mathbf{x}}_C^- \cdot \hat{\mathbf{s}}\right)\hat{\mathbf{s}}$$

$$\dot{\mathbf{x}}_1^+ = \dot{\mathbf{x}}_1^- + \frac{\mathbf{J}}{m_1} \qquad \dot{\mathbf{x}}_2^+ = \dot{\mathbf{x}}_2^- - \frac{\mathbf{J}}{m_2} \qquad \boldsymbol{\omega}_1^+ = \boldsymbol{\omega}_1^- + \frac{r_1}{I_1}\left(\hat{\mathbf{n}} \times \mathbf{J}\right) \qquad \boldsymbol{\omega}_2^+ = \boldsymbol{\omega}_2^- + \frac{r_2}{I_2}\left(\hat{\mathbf{n}} \times \mathbf{J}\right)$$

Some Additional Comments

- The three parameter hard-particle model (ε_N, ε_S^{SS}, and μ) is a “lumped parameter” model since the details of what occurs during the collision are not described.
- The collision parameters ε_N, ε_S^{SS}, and μ are not properties of a single material, but are the properties of the two interacting materials.
- Difficult to find and measure these model parameters.

References

Foerster, S.F., Louge, M.Y., Chang, H., and Allia, K., 1994, "Measurements of the collision properties of small spheres," *Physics of Fluids*, Vol. 6, pp. 1108-1115.

Goldsmith, W., 1960, *Impact: The Theory and Physical Behaviour of Colliding Solids*, Dover.

Gorham, D.A. and Kharaz, A.H., 2000, "The measurement of particle rebound characteristics," *Powder Technology*, Vol. 112, pp. 193 – 202.

Johnson, K.L., 1985, *Contact Mechanics*, Cambridge University Press.

Kruggel-Emden, H., Simsek, E., Rickelt, S., Wirtz, S., and Scherer, V., 2007, "Review and extension of normal force models," *Powder Technology*, Vol. 171, pp. 157 – 173.

Kuwabara, G. and Kono, K., 1987, "Restitution Coefficient in a Collision between Two Spheres," *Japanese Journal of Applied Physics*, Vol. 26, pp. 1230 – 1233.

Labous, L., Rosato, A.D., and Dave, R.N., 1997, "Measurements of collisional properties of spheres using high-speed video analysis," *Physical Review E*, Vol. 56, pp. 5717-5725.

Maw, N., Barber, J.R., and Fawcett, J.N., 1976, "The oblique impact of elastic spheres," *Wear*, Vol. 38, pp. 101 – 114.

Maw, N., Barber, J.R., and Fawcett, J.N., 1981, "The role of elastic tangential compliance in oblique impact," *Journal of Lubrication Technology*, Vol. 103, pp. 74 – 80.

Walton, O.R., 1993, "Numerical simulation of inelastic, frictional particle-particle interactions," in *Particulate Two-Phase Flow*, Chap. 25, M.C. Roco ed., Butterworth-Heinemann.