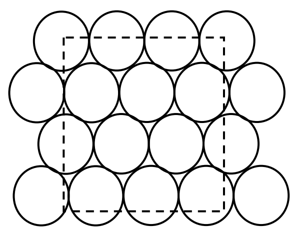
DEM Modeling: Lecture 07

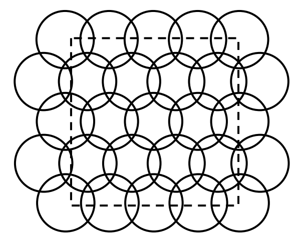
Normal Contact Force Models. Part II

Normal Contact Force Model
Effects of Soft Springs

- Excluded volume error
  - consider the solid fraction in a compressed system containing particles with stiff vs. soft springs



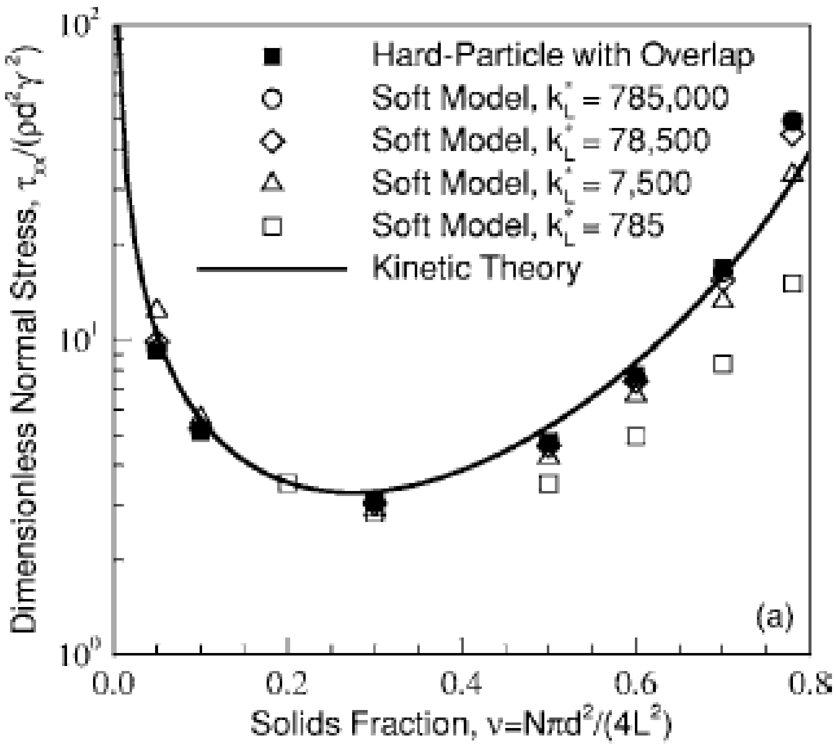
stiff springs
$\Rightarrow$ solid fraction < 1
(as expected)

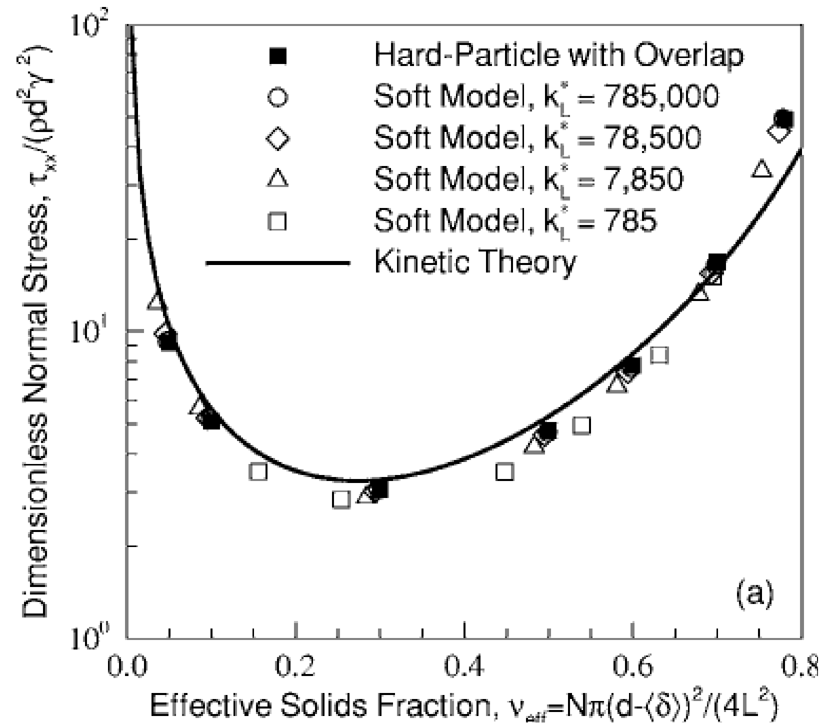


soft springs
$\Rightarrow$ solid fraction > 1!
(not realistic)

Normal Contact Force Model
Effects of Soft Springs…

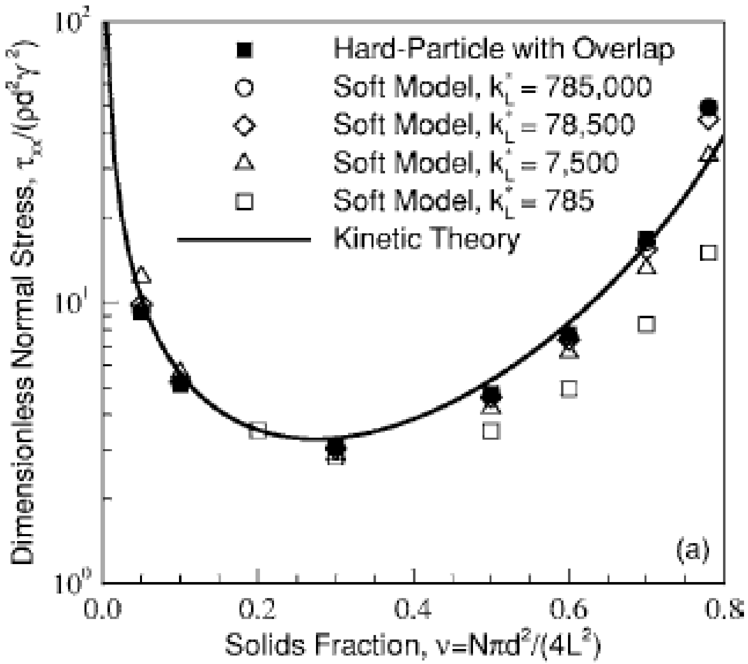
- Excluded volume error…
  - Ketterhagen *et al*. (2005) found that sheared systems with soft springs produce stresses similar to those that would be generated if smaller particles were used





Normal Contact Force Model Effects of Soft Springs…

2D shear simulations:



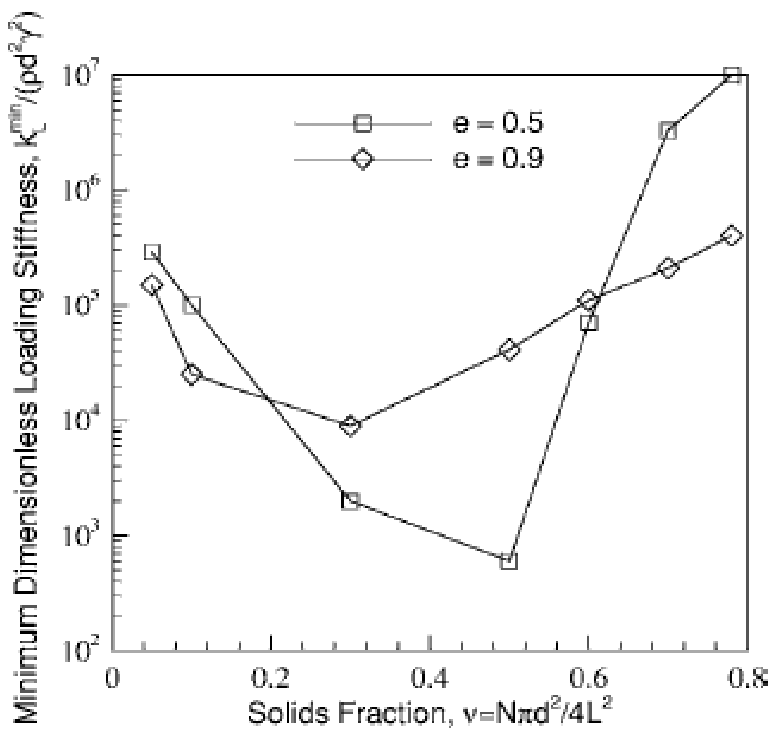


FIG. 10. Minimum dimensionless loading stiffness for the soft-particle model as a function of solids fraction for frictionless particles and e_N=0.5 and 0.9. Using stiffnesses larger than this minimum will ensure that stress results are within ±2.5% error from the respective asymptotic values.

as spring stiffness ↓
− at large solid fraction, stresses ↓
− at small solid fraction, stresses ↑

From Ketterhagen *et al*. (2005)

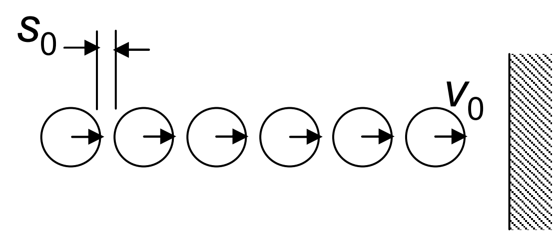
Normal Contact Force Model Effects of Soft Springs…

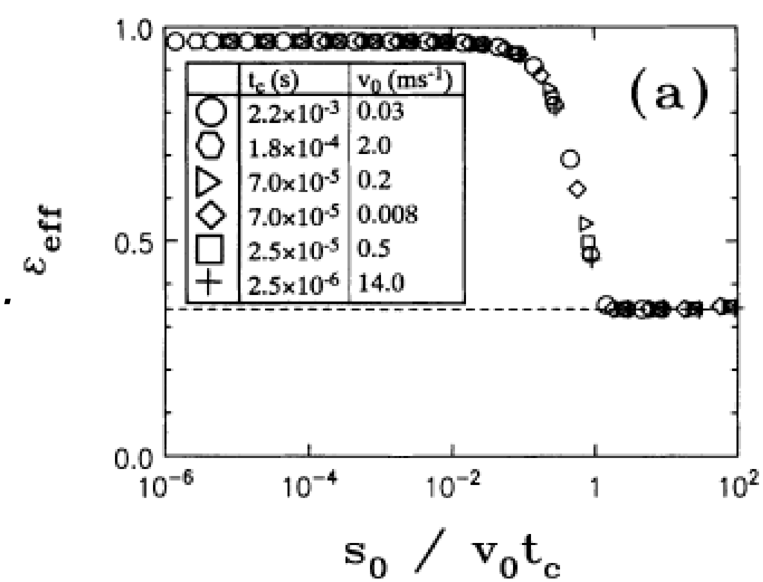
- Detachment effect
  - (Luding *et al.*,1994)
  - the effective coefficient of restitution of a system of particles depends upon the ratio of the time between collisions, $t_0 = s_0/v_0$ to the duration of an impact, T,

$$t_0/T \gg 1 \Rightarrow \varepsilon_{N,\text{eff}} \ll \varepsilon_N$$

$$t_0/T \ll N \Rightarrow \varepsilon_{N,\text{eff}} \approx 1$$

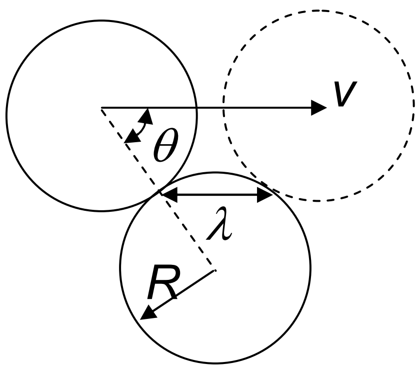
  - spring stiffnesses that are too soft (*i.e.* have too large of a T) may have a much smaller degree of energy dissipation than expected





Normal Contact Force Model Effects of Soft Springs…

- Brake efficiency failure (Schäfer and Wolf, 1995)

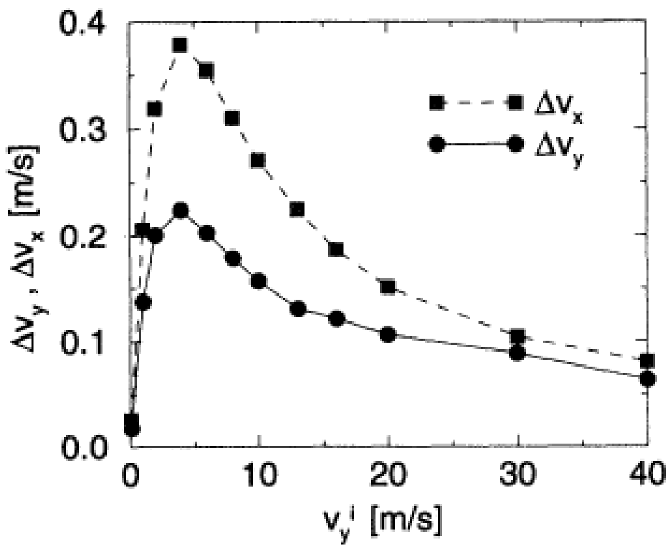


$\lambda = 2R\cos\theta$

if contact is stiff, then $t_{\text{contact}} \approx T$ and $\Delta v \propto v$

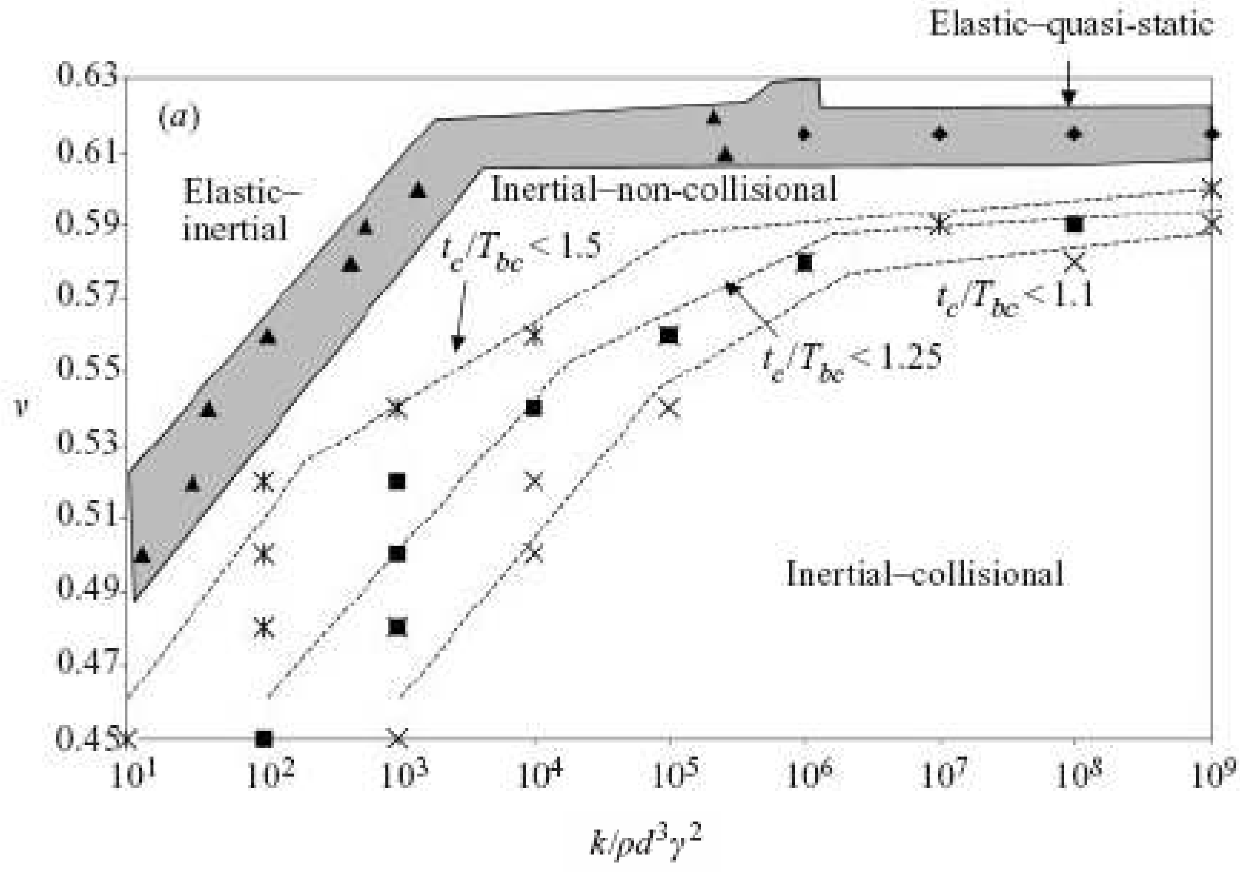
if contact is soft, then $t_{\text{contact}} \approx \lambda/v$ and $\Delta v \propto 1/v$

$$T = \tau \Rightarrow \pi\sqrt{\frac{m'}{k}} \approx \frac{\lambda}{v_{\text{crit}}} \Rightarrow \boxed{v_{\text{crit}} \approx \frac{2R}{\pi}\cos\theta\sqrt{\frac{k}{m'}}}$$



$\Rightarrow$ for contacts with $v > v_{\text{crit}}$, the "braking efficiency" decreases with increasing impact speed; particle rebound response is significantly altered

Normal Contact Force Model Effects of Soft Springs…



Stress scaling varies depending on the dimensionless stiffness and solid fraction

elastic-quasi-static:

$$\frac{\tau}{kd} = f(\nu, \mu, \varepsilon)$$

inertial-collisional:

$$\frac{\tau}{\rho d^2 \gamma^2} = f(\nu, \mu, \varepsilon)$$

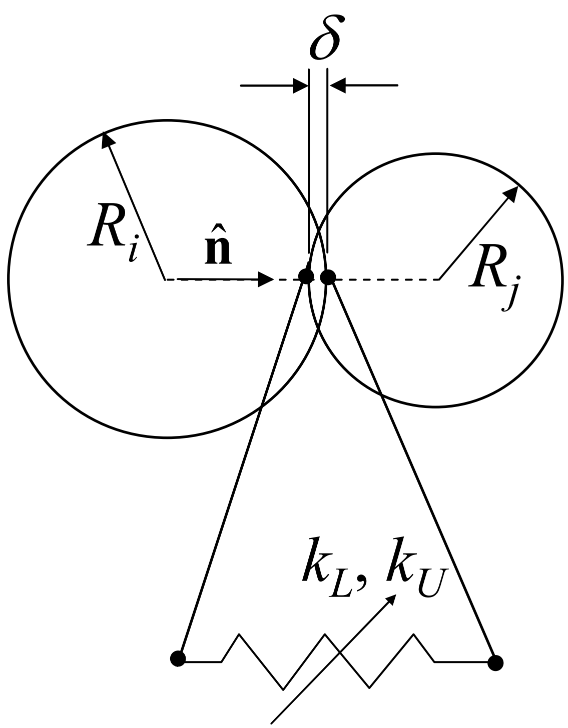
elastic-inertial:

$$\frac{\tau}{\rho d^2 \gamma^2} \text{ or } \frac{\tau}{kd} = f\left(\frac{k}{\rho d^3 \gamma^2} \nu, \mu, \varepsilon\right)$$

From Campbell (2002)

Normal Contact Force Model Hysteretic Linear Spring

- First proposed by Walton and Braun (1986)
- Widely used



$$\mathbf{F}_i = \begin{cases} -k_L \delta \hat{\mathbf{n}} & \delta \geq \delta_{\text{max}} \\ -k_U \left(\delta - \delta_{\text{res}}\right)\hat{\mathbf{n}} & \delta_{\text{res}} < \delta < \delta_{\text{max}} \\ \mathbf{0} & 0 \leq \delta \leq \delta_{\text{res}} \end{cases}$$

k_L ≡ loading spring stiffness
k_U ≡ unloading spring stiffness ($k_U \geq k_L$)
δ_{res} ≡ residual overlap
δ_{max} ≡ maximum overlap during contact

- energy dissipation due to spring force hysteresis
- contact force is continuous
- energy dissipation is position dependent
- relatively simple model to implement, but need to retain history of δ_{max}

Normal Contact Force Model Hysteretic Linear Spring…

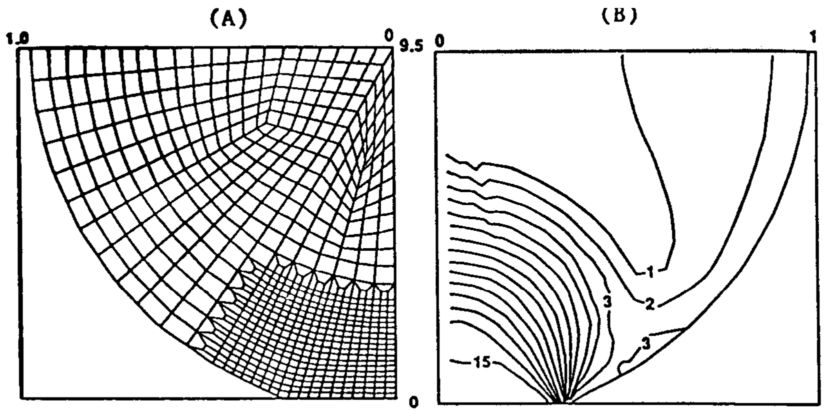


FIGURE 25-4 *a) Representative zoning used in finite element calculations of a hemisphere impinging on a rigid wall (Walton and Brandeis, 1984), and b) Calculated equal pressure contours using elastic-perfectly-plastic constitutive model.*

- The hysteretic linear spring model mimics elastic-perfectly plastic, quasi-static FEM model results.
- The hysteretic stiffnesses (k_L, k_U) model the strain hardening of the material due to plastic deformation.
- The residual overlap (δ_{res}) represents the permanent plastic deformation of the contact.

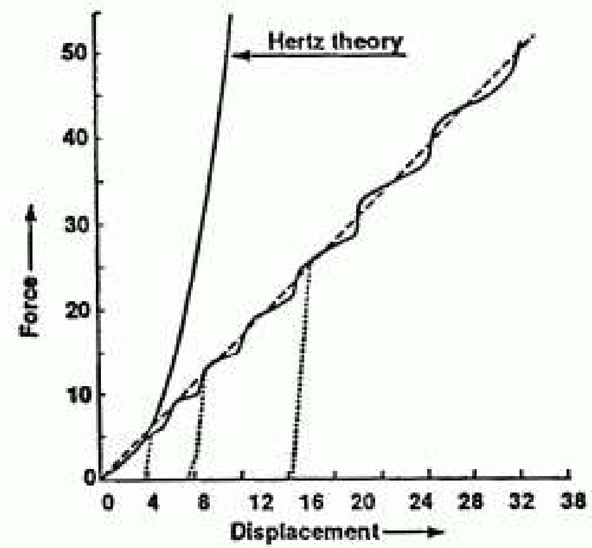
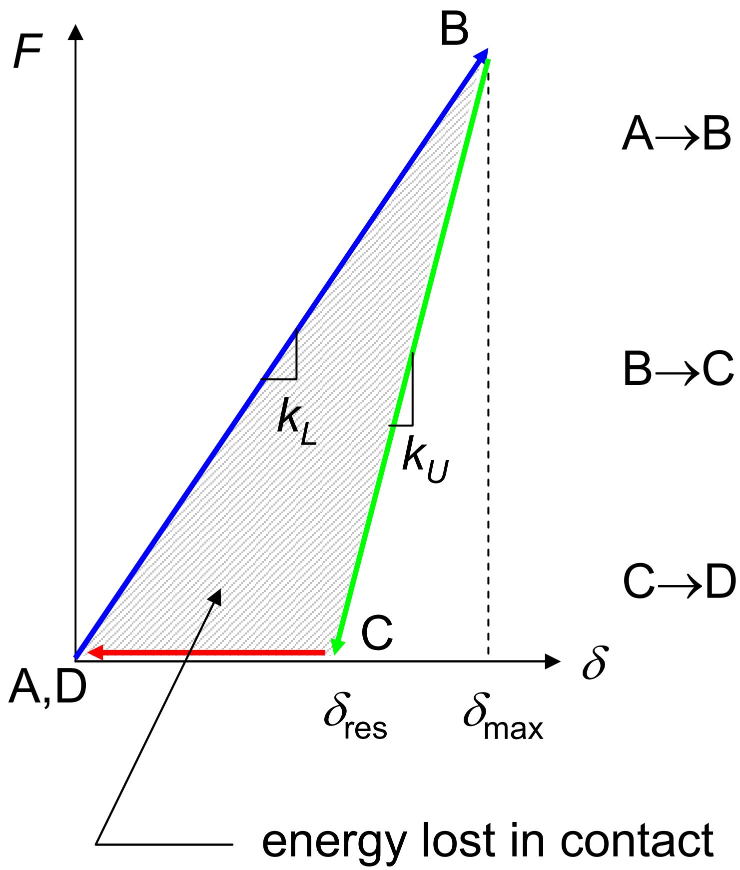


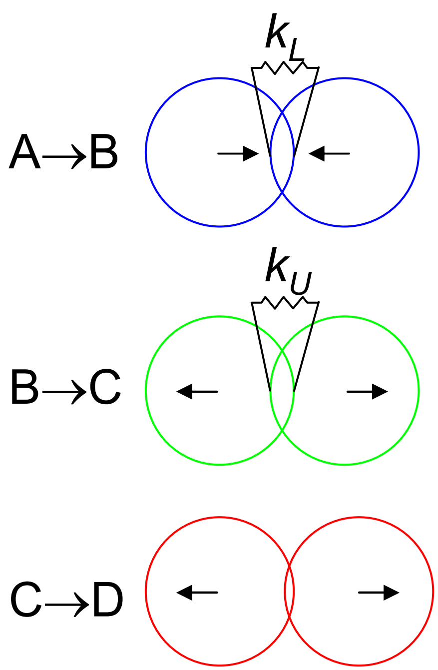
FIGURE 25-5. *Loading and unloading force-displacement behavior for elastic-plastic spheres during quasi static normal displacement as calculated using NIKE2D finite element model (Walton and Brandeis, 1984)*

From: Walton, O.R. (1993)

Normal Contact Force Model Hysteretic Linear Spring…

A simple example:





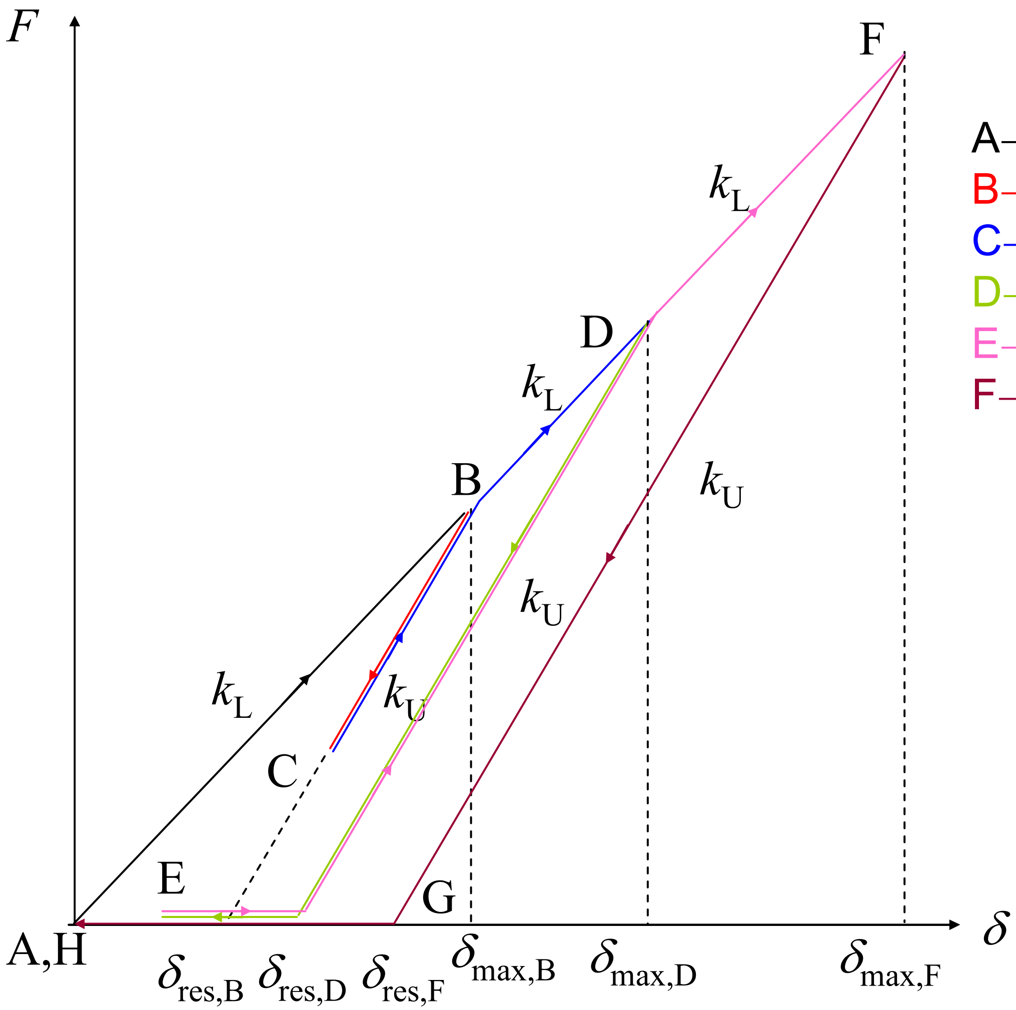
$$k_L \delta_{\max} = k_U \left(\delta_{\max} - \delta_{\mathrm{res}} \right)$$

$$\Rightarrow \delta_{\mathrm{res}} = \delta_{\max} \left(1 - \frac{k_L}{k_U} \right)$$

Note: After full unloading (after reaching pt. D), the residual overlap is "forgotten" for the contact. The "plastic deformation" for the contact only exists during the contact.

Normal Contact Force Model
Hysteretic Linear Spring…

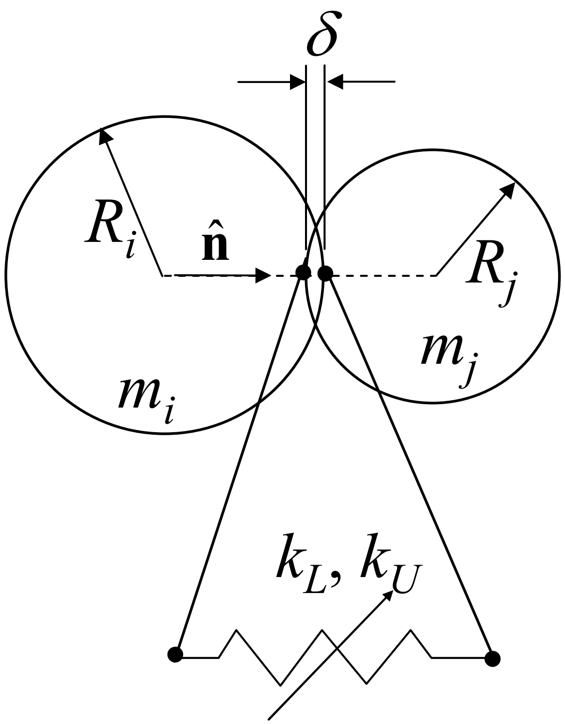
A more complex example:



A→B:	initial loading
B→C:	partial unloading
C→B→D:	reloading beyond $\delta_{max,B}$
D→E:	partial unloading beyond $\delta_{res,D}$
E→D→F:	reloading beyond $\delta_{max,D}$
F→G→H:	full unloading

Normal Contact Force Model Hysteretic Linear Spring…

For a two particle contact (derivation is left as an exercise):



$$\delta_{\max} = \dot{\delta}_0 \sqrt{\frac{m'}{k_L}}$$

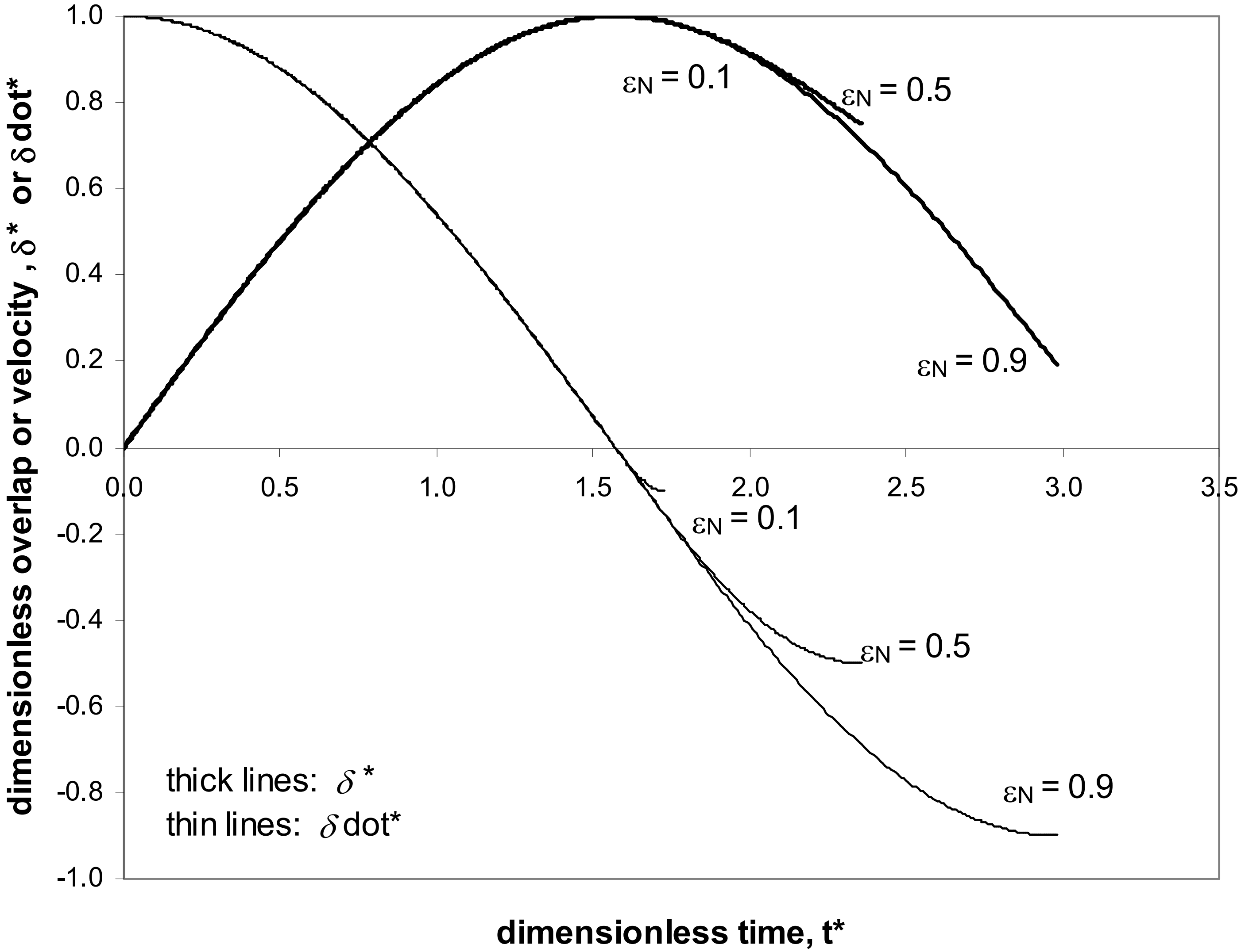
$$\varepsilon_N = \sqrt{\frac{k_L}{k_U}}$$

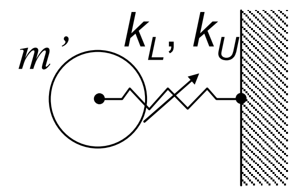
$$T = \sqrt{\frac{m'}{k_L}} \frac{\pi}{2} \left(\varepsilon_N + 1 \right)$$

$$\delta_{\text{res}} = \dot{\delta}_0 \sqrt{\frac{m'}{k_L}} \left(1 - \varepsilon_N^2 \right)$$

$\dot{\delta}_0$ ≡ relative impact speed
m' ≡ effective mass (= $(m_i^{-1} + m_j^{-1})^{-1}$)
ε_N ≡ normal coefficient of restitution
T ≡ contact duration

Normal Contact Force Model
Hysteretic Linear Spring…



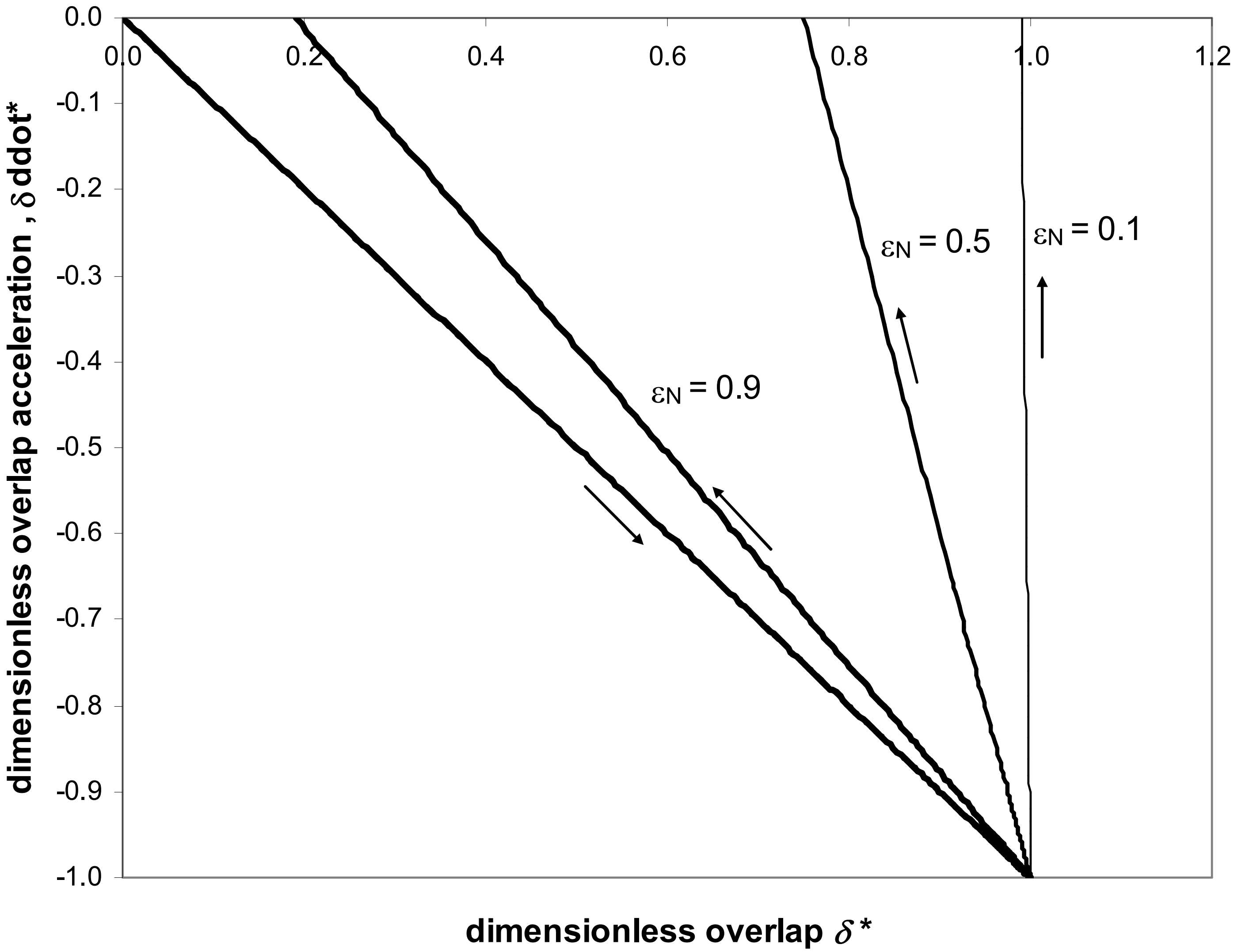


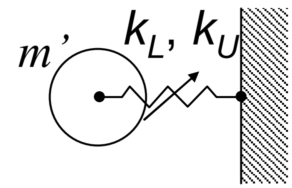
$$\delta^* \equiv \frac{\delta}{\dot{\delta}_0}\sqrt{\frac{k}{m'}}$$

$$\dot{\delta}^* \equiv \frac{\dot{\delta}}{\dot{\delta}_0}$$

$$t^* \equiv t\sqrt{\frac{k}{m'}}$$

Normal Contact Force Model
Hysteretic Linear Spring…





$$\ddot{\delta}^* \equiv \frac{\ddot{\delta}}{\dot{\delta}_0}\sqrt{\frac{m'}{k}}$$

$$\delta^* \equiv \frac{\delta}{\dot{\delta}_0}\sqrt{\frac{k}{m'}}$$

$$t^* \equiv t\sqrt{\frac{k}{m'}}$$

Normal Contact Force Model Hysteretic Linear Spring…

- Some observations
  - the contact force is continuous at the start and end of the contact
  - the contact force is always repulsive
  - the loading portion of the contact is the same regardless of the coefficient of restitution
  - energy dissipation is due to the hysteresis in the overlap

Normal Contact Force Model Hysteretic Linear Spring…

- Some observations…
  - coefficient of restitution is independent of impact speed (in real collisions $\varepsilon_N \downarrow$ as $\dot{\delta}_0 \uparrow$)
  - contact duration $\uparrow$ as $k_L \downarrow$, $m' \uparrow$, and $\varepsilon_N \uparrow$
    - coefficient of restitution dependence is opposite that for the damped linear spring model
    - contact duration is independent of impact speed (in real collisions, contact duration $\downarrow$ as impact speed $\uparrow$)
  - maximum overlap $\uparrow$ as $\dot{\delta}_0 \uparrow$, $m' \uparrow$, $k_L \downarrow$
    - larger overlaps make the geometrically rigid particle assumption less accurate and can cause modeling errors due to excluded volume effects
    - unlike the damped linear spring model, coefficient of restitution does not play a role

Normal Contact Force Model Hysteretic Linear Spring…

- The unloading spring stiffness (k_U) may be found from the loading spring stiffness (k_L) and the coefficient of restitution (ε_N):

$$\varepsilon_N = \sqrt{\frac{k_L}{k_U}}$$

- Method for determining the loading spring stiffness is not widely agreed upon
  - consider three methods, all of which set particular contact parameters equal to those found using a Hertzian contact model
    - maximum overlap
    - contact duration
    - maximum strain energy

Normal Contact Force Model Hysteretic Linear Spring…

– equivalent maximum overlap, δ_{max}

HLS model: $\delta_{\max} = \dot{\delta}_0 \sqrt{\frac{m'}{k_L}}$

Hertzian spring model: $\delta_{\max} = \left(\frac{15}{16} \frac{m'}{R'^{1/2} E'} \dot{\delta}_0^2 \right)^{2/5}$

$$\therefore k_{L,\text{overlap}} \approx 1.053 \left(\dot{\delta}_0 m'^{1/2} R' E'^2 \right)^{2/5}$$

Normal Contact Force Model Hysteretic Linear Spring…

– equivalent contact duration, *T*

HLS model: $T = \sqrt{\frac{m'}{k_L}} \frac{\pi}{2}\left(\varepsilon_N + 1\right)$

Hertzian spring model: $T \approx 2.870\left(\frac{m'^2}{R'E'^2\dot{\delta}_0}\right)^{1/5}$

$$\therefore k_{L,\text{duration}} = 0.2996\left(\dot{\delta}_0 m'^{1/2} R'E'^2\right)^{2/5}\left(\varepsilon_N + 1\right)^2$$

Normal Contact Force Model Hysteretic Linear Spring…

– maximum strain energy, $SE_{\max}$

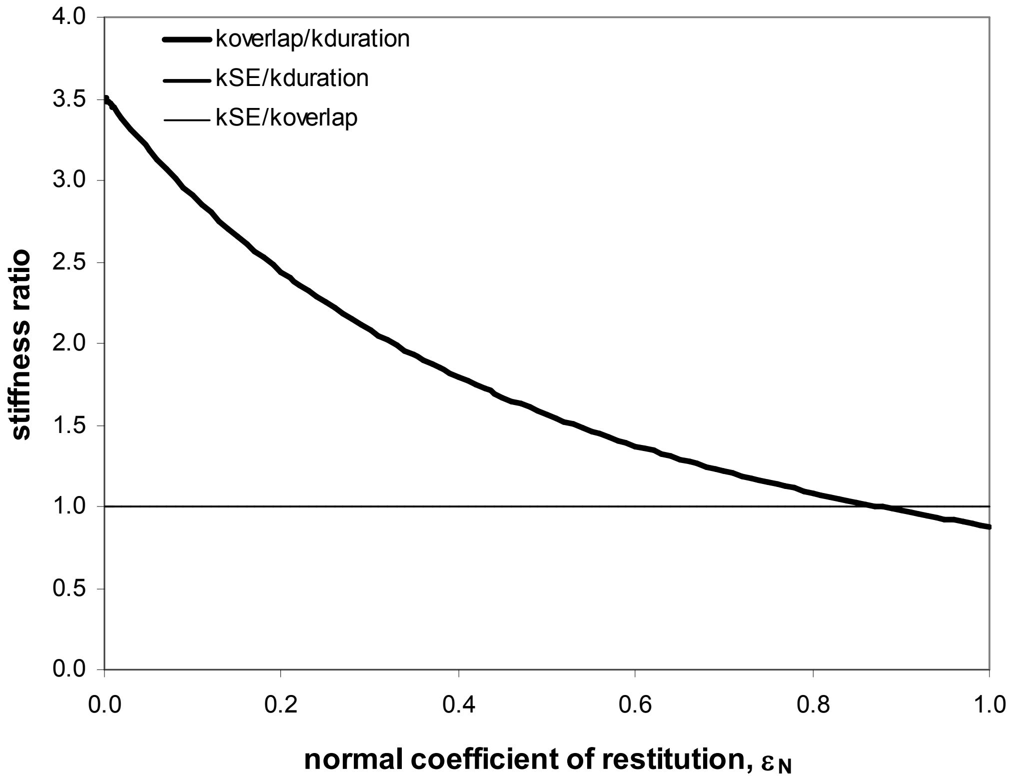
HLS model: $SE_{\max} = \frac{1}{2} k_L \delta_{\max}^2$

Hertzian spring model: $SE_{\max} = \frac{2}{5} k_{Hz} \delta_{\max}^{5/2}$ where $k_{Hz} = \frac{4}{3} R'^{1/2} E'$

$$\delta_{\max} = \left(\frac{15}{16} \frac{m'}{R'^{1/2} E'} \dot{\delta}_0^2 \right)^{2/5}$$

$$\therefore k_{L,\mathrm{SE}} \approx 1.053 \left(\dot{\delta}_0 m'^{1/2} R' E'^2 \right)^{2/5}$$

Normal Contact Force Model Hysteretic Linear Spring…



$$\frac{k_{L,\text{overlap}}}{k_{L,\text{duration}}} = \frac{k_{L,SE}}{k_{L,\text{duration}}} = \frac{3.515}{(\varepsilon_N + 1)^2}$$

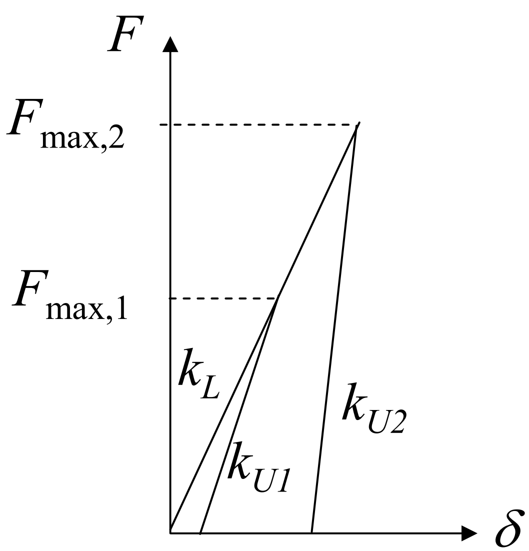
- For example: two 3.18 mm diam. soda lime glass spheres (ρ = 2500 kg/m^3, ν = 0.22, E = 71 GPa) impacting at 1 m/s (ε_N = 0.97; Foerster *et al*., 1994):
  - max overlap/eff. diameter = 0.2% $\Rightarrow$ k_{overlap} = 2.02 MN/m
  - contact duration = 9.51 μs $\Rightarrow$ k_{duration} = 2.23 MN/m
  - max strain energy = 10.5 μJ $\Rightarrow$ k_{SE} = 2.02 MN/m

Normal Contact Force Model
Hysteretic Linear Spring…

- With a constant k_U, the coefficient of restitution remains constant regardless of impact speed
- A variable coefficient of restitution may be modeled using a variable unloading stiffness (Walton and Braun, 1986).

$$k_U = k_L + SF_{\max}$$

where S is a constant and $F_{\max}$ is the maximum force achieved before unloading. The constant S can be determined empirically from experimental data. (S ~ $10^4 - 10^6$ m^{-1}).



$$\Rightarrow \varepsilon_N = \sqrt{\frac{1}{1 + S\dot{\delta}_0\sqrt{\frac{m'}{k_L}}}}$$

$$\therefore S = \frac{1}{\dot{\delta}_0}\sqrt{\frac{k_L}{m'}}\left(\frac{1}{\varepsilon_N^2} - 1\right)$$

$$T = \sqrt{\frac{m'}{k_L}}\frac{\pi}{2}\left(\varepsilon_N + 1\right)$$

Normal Contact Force Model Hysteretic Linear Spring…

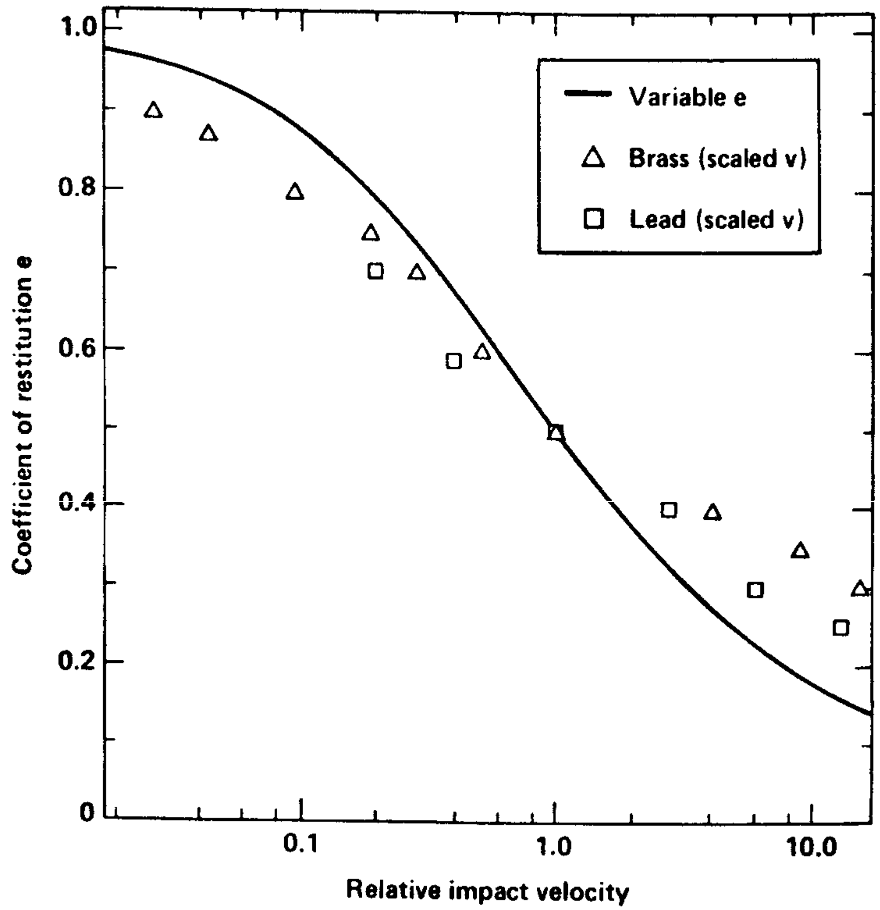


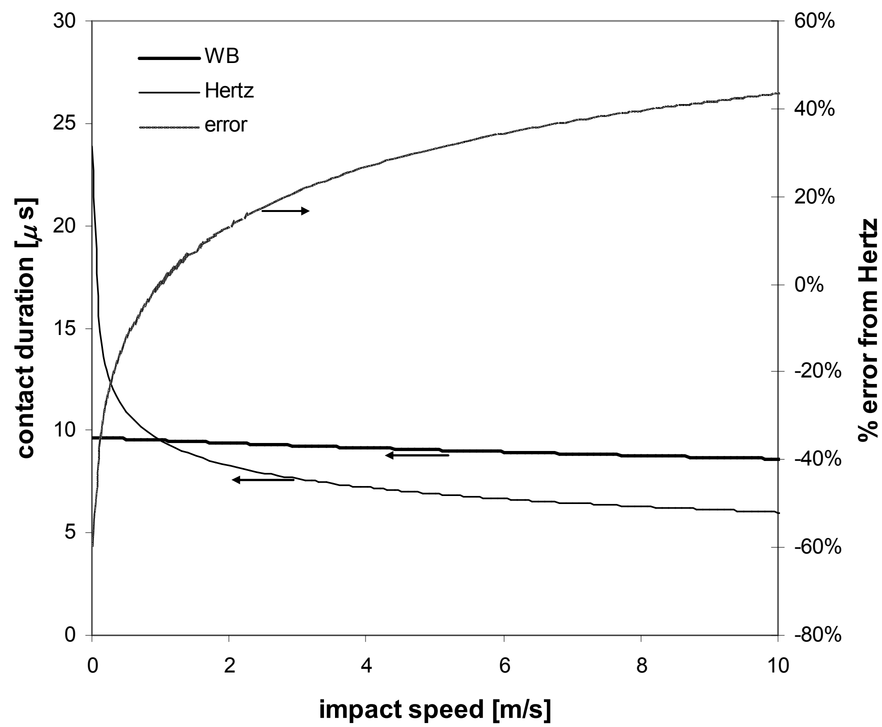
Fig. 2. Coefficient of restitution given by variable e model (Eqs. 2 to 5) and obtained in impact tests with identical spheres of brass and lead.[18] Velocities scaled so that 1 corresponds to $e = 1/2$.

From Walton and Braun (1986)

Normal Contact Force Model Hysteretic Linear Spring…

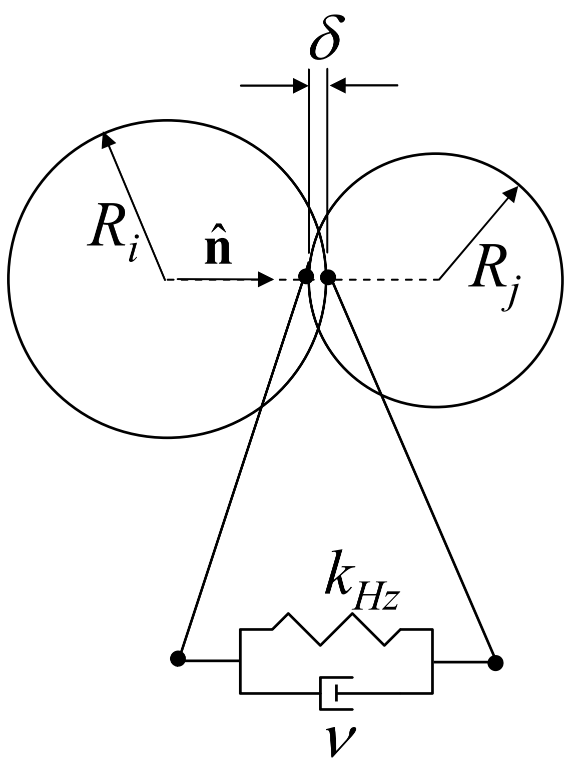
- Some observations…
  - coefficient of restitution ↓ as impact speed ↑
    - matches experimental values reasonably well
  - contact duration ↓ as impact speed ↑
    - very poor match to experimental data

- For example:
  - two 3.18 mm diam. spheres
  - soda lime glass: ρ = 2500 kg/m^3, ν = 0.22, E = 71 GPa
  - match k_{dur} and S at 1 m/s (ε_N = 0.97)
  - $k_{duration}$ = 2.23 MN/m
  - S = 2.04*10^4 m^{-1}



Normal Contact Force Model Damped Hertzian Spring

- Taguchi (1992); Ristow (1992); Pöschel (1993); Lee and Hermann (1993); Zhou *et al*. (1999)



$$\mathbf{F}_i = \left(-k_{Hz}\delta^{3/2} + \nu\dot{\delta}\right)\hat{\mathbf{n}}$$

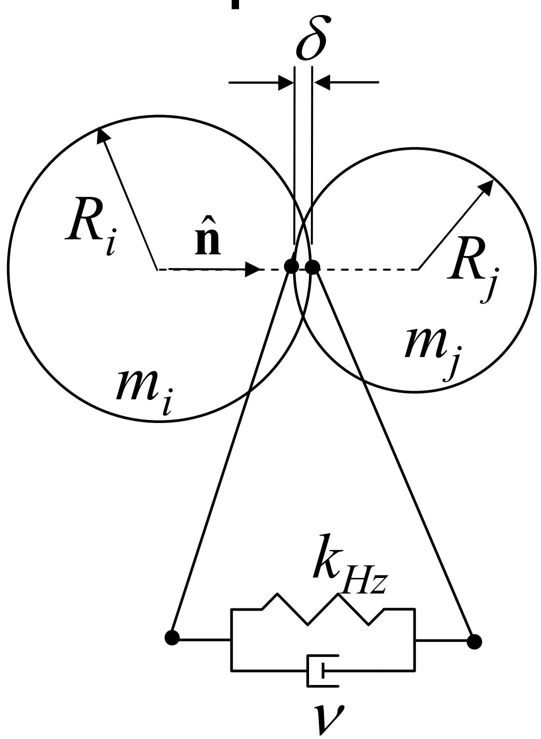
k_{Hz} ≡ Hertzian spring stiffness = $4/3R'^{1/2}E'$

ν ≡ damping coefficient

- spring force is consistent with Hertz model
- dashpot added (in an ad hoc fashion) to provide energy dissipation
- contact force is discontinuous at start/end of contact due to damping force
- energy dissipation is velocity dependent
- simple model to implement

Normal Contact Force Model Damped Hertzian Spring…

For a two particle contact (derivation is left as an exercise):



$$\ddot{\delta}^* + \nu^* \dot{\delta}^* + \delta^{*3/2} = 0$$

$$\nu^* \equiv \frac{\nu}{\left(m'^{3/2} \dot{\delta}_0^{1/2} k_{Hz}\right)^{2/5}}$$

$$t^* \equiv t\left(\frac{k_{Hz}\dot{\delta}_0^{1/2}}{m'}\right)^{2/5}$$

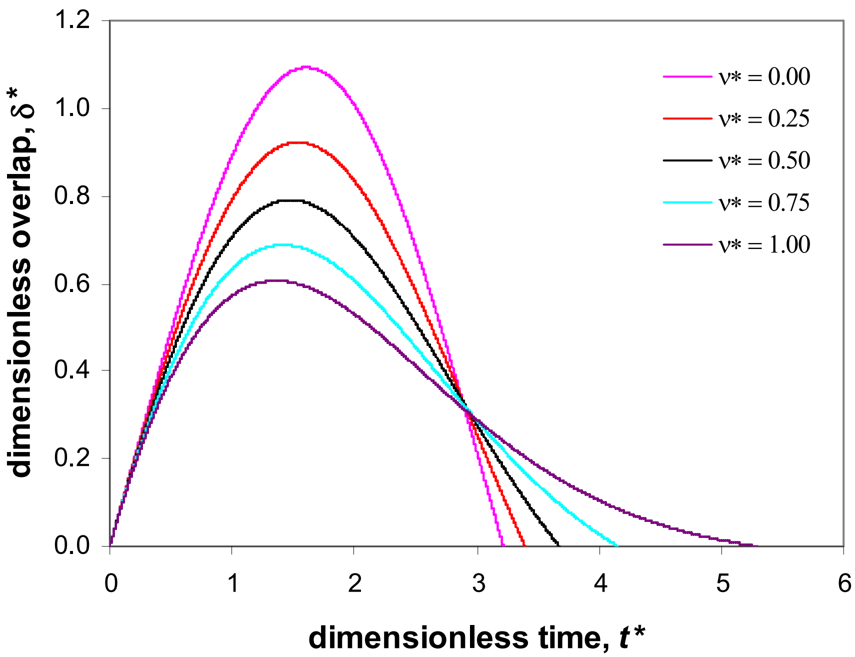
$$\delta^* \equiv \delta\left(\frac{k_{Hz}}{\dot{\delta}_0^2 m'}\right)^{2/5}$$

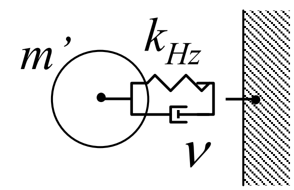
$$\dot{\delta}^* = \frac{\dot{\delta}}{\dot{\delta}_0}$$

$$\ddot{\delta}^* \equiv \ddot{\delta}\left(\frac{m'}{\dot{\delta}_0^3 k_{Hz}}\right)^{2/5}$$

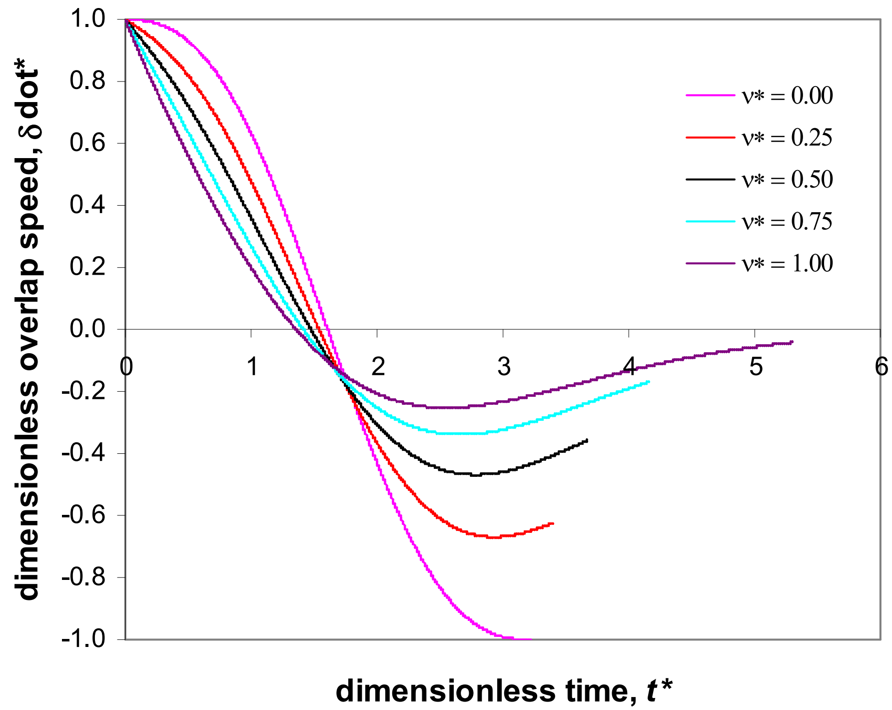
$\dot{\delta}_0$ ≡ relative impact speed
m' ≡ effective mass (= $(m_i^{-1} + m_j^{-1})^{-1}$)
ν ≡ damping coefficient
k_{Hz} ≡ Hertz spring stiffness = $4/3 R'^{1/2} E'$

Normal Contact Force Model
Damped Hertzian Spring…





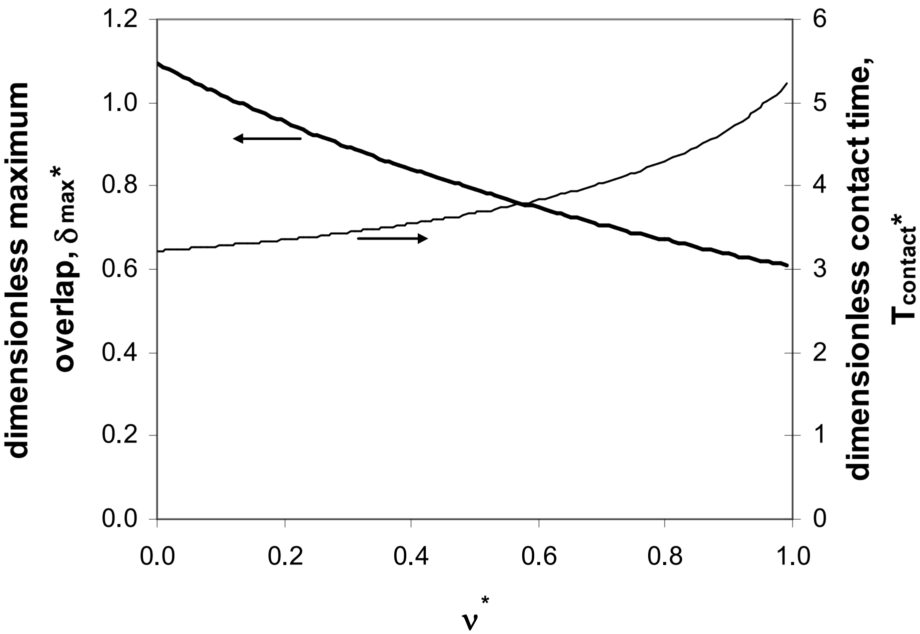
$\nu^* = 0.00 \Rightarrow \varepsilon_N = 1.00$
$\nu^* = 0.25 \Rightarrow \varepsilon_N = 0.62$
$\nu^* = 0.50 \Rightarrow \varepsilon_N = 0.37$
$\nu^* = 0.75 \Rightarrow \varepsilon_N = 0.17$
$\nu^* = 1.00 \Rightarrow \varepsilon_N = 0.04$

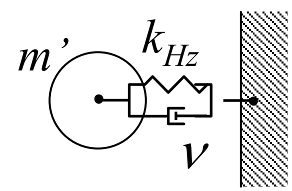


$$\nu^* \equiv \frac{\nu}{\left(m'^{3/2}\dot{\delta}_0^{1/2}k_{Hz}\right)^{2/5}} \qquad t^* \equiv t\left(\frac{k_{Hz}\dot{\delta}_0^{1/2}}{m'}\right)^{2/5} \qquad \delta^* \equiv \delta\left(\frac{k_{Hz}}{\dot{\delta}_0^2 m'}\right)^{2/5}$$

$$\dot{\delta}^* = \frac{\dot{\delta}}{\dot{\delta}_0} \qquad \ddot{\delta}^* \equiv \ddot{\delta}\left(\frac{m'}{\dot{\delta}_0^3 k_{Hz}}\right)^{2/5}$$

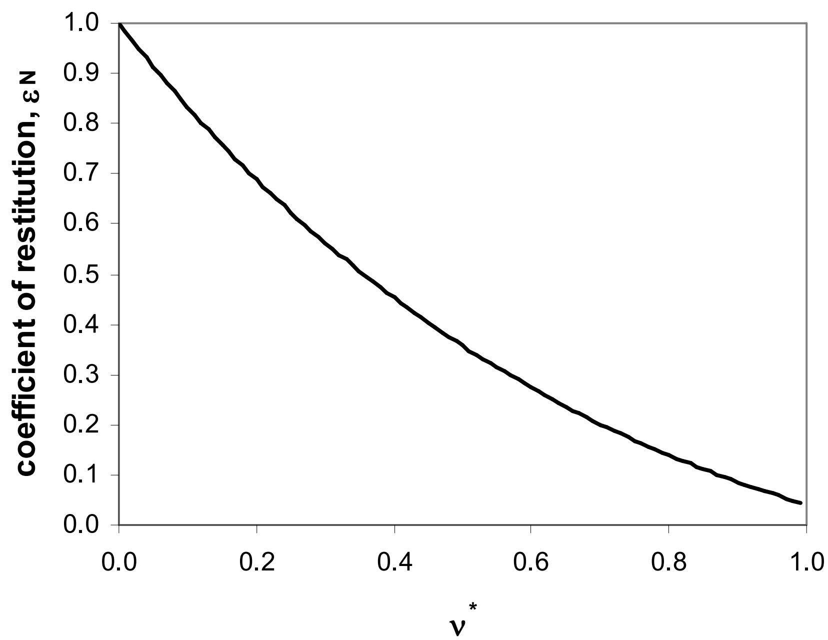
Normal Contact Force Model Damped Hertzian Spring…



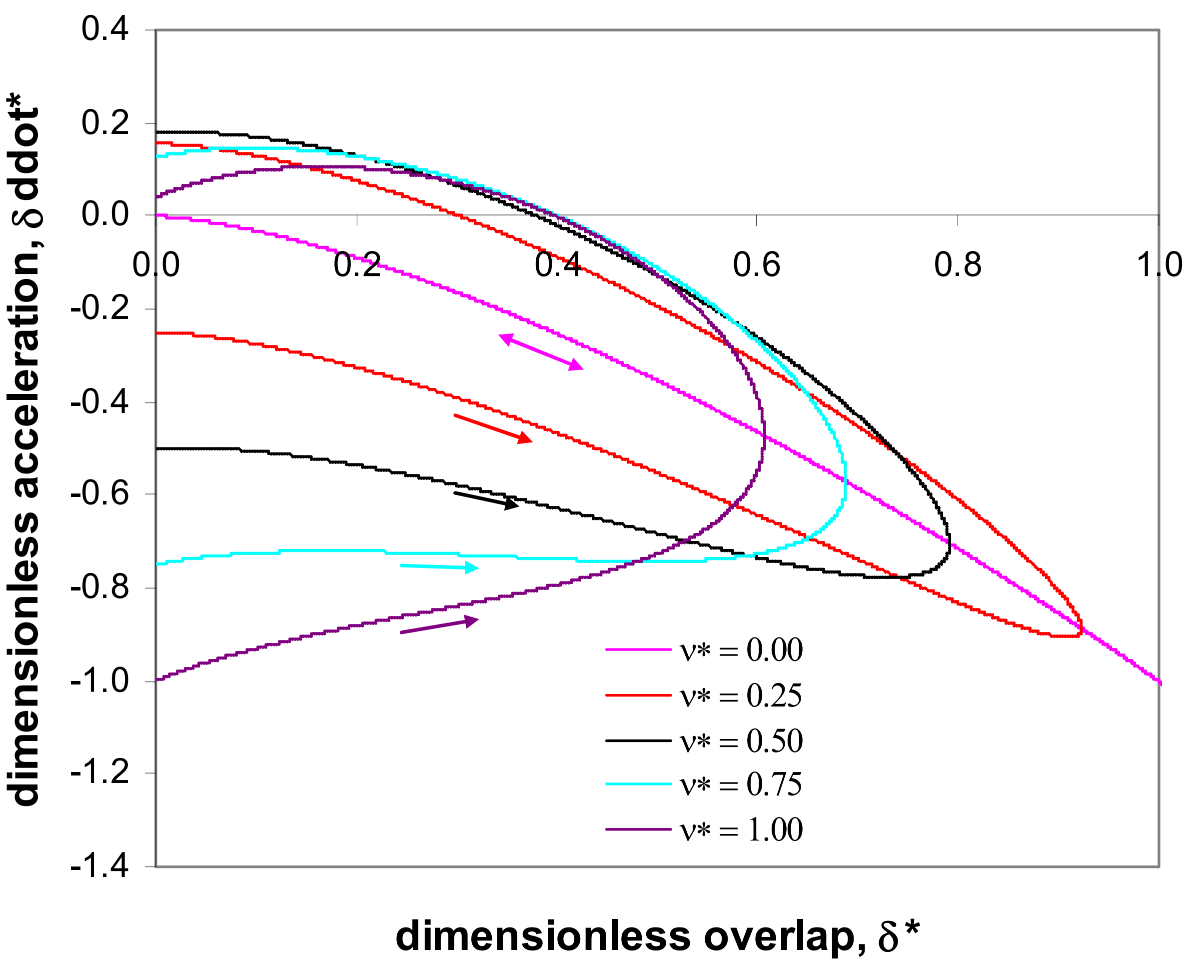


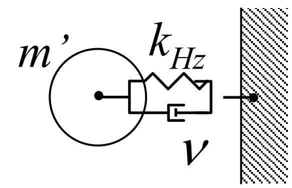
$$\nu^* \equiv \frac{\nu}{\left(m'^{3/2}\dot{\delta}_0^{1/2}k_{Hz}\right)^{2/5}} \quad t^* \equiv t\left(\frac{k_{Hz}\dot{\delta}_0^{1/2}}{m'}\right)^{2/5} \quad \delta^* \equiv \delta\left(\frac{k_{Hz}}{\dot{\delta}_0^2 m'}\right)^{2/5}$$

$$\dot{\delta}^* = \frac{\dot{\delta}}{\dot{\delta}_0} \quad \ddot{\delta}^* \equiv \ddot{\delta}\left(\frac{m'}{\dot{\delta}_0^3 k_{Hz}}\right)^{2/5}$$



Normal Contact Force Model
Damped Hertzian Spring…





$\nu^* = 0.00 \Rightarrow \varepsilon_N = 1.00$
$\nu^* = 0.25 \Rightarrow \varepsilon_N = 0.62$
$\nu^* = 0.50 \Rightarrow \varepsilon_N = 0.37$
$\nu^* = 0.75 \Rightarrow \varepsilon_N = 0.17$
$\nu^* = 1.00 \Rightarrow \varepsilon_N = 0.04$

$$\nu^* \equiv \frac{\nu}{\left(m'^{3/2}\dot{\delta}_0^{1/2}k_{Hz}\right)^{2/5}} \qquad t^* \equiv t\left(\frac{k_{Hz}\dot{\delta}_0^{1/2}}{m'}\right)^{2/5}$$

$$\delta^* \equiv \delta\left(\frac{k_{Hz}}{\dot{\delta}_0^2 m'}\right)^{2/5} \qquad \dot{\delta}^* = \frac{\dot{\delta}}{\dot{\delta}_0}$$

$$\ddot{\delta}^* \equiv \ddot{\delta}\left(\frac{m'}{\dot{\delta}_0^3 k_{Hz}}\right)^{2/5}$$

Normal Contact Force Model Damped Hertzian Spring…

- Some observations
  - the contact force is discontinuous at the start and end of the contact due to the viscous damping force (real contact forces are continuous)
  - the contact force is cohesive toward the end of the impact (real contact forces are always repulsive for cohesionless systems)
  - $\varepsilon_N \uparrow$ as impact speed $\uparrow$ (just the opposite in real collisions)
    - $\nu^* \downarrow$ as δdot $\uparrow$
    - $\varepsilon_N \rightarrow 0$ as δdot $\rightarrow 0$!
  - contact duration $\downarrow$ as impact speed $\uparrow$ (consistent with real data)

References

- Buchholtz, V. and Pöschel, T., 1994, "Numerical investigations of the evolution of sandpiles," *Physica A*, Vol. 202, Nos. 3-4, pp. 390 – 401.
- Campbell, C.S., 2002, "Granular shear flows at the elastic limit," *Journal of Fluid Mechanics*, Vol. 465, pp. 261 – 291.
- Corkum, B.T. and Ting, J.M., 1986, *The discrete element method in geotechnical engineering*, Publication 86-11, Department of Civil Engineering, University of Toronto, ISBN 0-7727-7086-7.
- Cundall, P.A. and Strack, O.D.L., 1979, "A discrete numerical model for granular assemblies," *Géotechnique*, Vol. 29, No. 1, pp. 47 – 65.
- Dury, C.M. and Ristow, G.H., 1997, "Radial segregation in a two-dimensional rotating drum," *Journal de Physique I*, Vol. 7, No. 5, pp. 737 – 745.
- Foerster, S.F., Louge, M.Y., Chang, H., and Allis, K., 1994, "Measurements of the collision properties of small spheres," *Physics of Fluids*, Vol. 6, No. 3, pp. 1108 – 1115.
- Goddard, J.D., 1990, "Nonlinear elasticity and pressure-dependent wave speeds in granular media," *Proceedings of the Royal Society London A: Mathematical and Physical Sciences*, Vol. 430, No. 1878, pp. 105 – 131.
- Goldsmith, W., 1960, *Impact: The Theory and Physical Behaviour of Colliding Solids*, Dover.
- Hertz, H., 1882, "Über die Berührung fester elastischer Körper," *J. reine und angewandte Mathematik*, Vol. 92, pp. 156 – 171.
- Johnson, K.L., 1985, *Contact Mechanics*, Cambridge University Press.
- Ketterhagen, W.R., Curtis, J.S., and Wassgren, C.R., 2005, "Stress results from two-dimensional granular shear flow simulations," *Physical Review E*, Vol. 71, Art. 061307.
- Kruggel-Emden, H., Simsek, E., Rickelt, S., Wirtz, S., and Scherer, V., 2007, "Review and extension of normal force models for the Discrete Element Method," *Powder Technology*, Vol. 171, pp. 157 – 173.
- Kuwabara, G. and Kono, K., 1987, "Restitution coefficient in a collision between two spheres," *Jpn. J.Appl. Phys.*, Vol. 26, pp. 1230 – 1233.
- Labous, L., Rosato, A.D., Dave, R.N., 1997, "Measurement of collisional properties of spheres using high-speed video analysis," *Physical Review E*, Vol. 56, pp. 5717 – 5725.
- Lan, Y. and Rosato, A.D., 1995, "Macroscopic behavior of vibrating beds of smooth inelastic spheres," *Physics of Fluids*, Vol. 7, No. 8, pp. 1818 – 1831.
- Lan, Y. and Rosato, A.D., 1997, "Convection related phenomena in granular dynamics simulations of vibrated beds," *Physics of Fluids*, Vol. 9, No 12, pp. 3615 – 3624.
- Luding, S., Clément, E., Blumen, A., Rajchenback, J., and Duran, J., 1994, "Anomalous energy dissipation in molecular-dynamics simulation of grains: The detachment effect," *Physical Review E*, Vol. 50, No. 5, pp. 4113 – 4124.
- Mullier, M., Tüzün, U., and Walton, O.R., 1991, "A single-particle friction cell for measuring contact frictional properties of granular materials," *Powder Technology*, Vol. 65, pp. 61 – 74.
- Schäfer, J. and Wolf, D.E., 1995, "Bistability in simulated granular flow along corrugated walls," *Physical Review E*, Vol. 51, No. 6, pp. 6154 – 6157.
- Stevens, A.B. and Hrenya, C.M., 2005, "Comparison of soft-sphere models to measurements of collision properties during normal impacts," *Powder Technology*, Vol. 154, pp. 99 – 109.
- Walton, O.R., 1993, "Numerical simulation of inelastic, frictional particle-particle interactions," in *Particulate Two-Phase Flow*, M.C. Roco, ed., Chap. 25, pp. 884 – 911, Butterworth-Heinemann.